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Zafar Projected Differential Transform and Laplace Projected Differential Transform methods as exact solution methods for Klein-Gordon equations

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Abstract

In this study, we propose two innovative numerical methods for solving Klein-Gordon equations: the Zafar Projected Differential Transform Method (ZPDTM) and the Laplace Projected Differential Transform Method (LPDTM). By integrating the Zafar and Laplace transforms respectively with the Projected Differential Transform Method, these approaches offer improved computational efficiency and enhanced solution accuracy. The performance of both methods is demonstrated through their application to linear and nonlinear forms of the Klein-Gordon equation, showing strong agreement with exact solutions and reduced computational overhead. These results highlight the versatility and reliability of ZPDTM and LPDTM in addressing complex differential models encountered in physics and engineering.

Keywords: Zafar Transform, Laplace Transform, Projected Differential Transform, Klein-Gordon equations, Exact solution.

INTRODUCTION

In the realm of scientific modeling, it is common for a preceding model aimed at elucidating a specific phenomenon to be surpassed by a more robust alternative [1-5]. This occurrence is frequently observed, wherein the previous model may find applicability in addressing real-world problems vastly different from its original context [4-8]. A notable example of this scenario is found in the development of the Klein–Gordon equation, first formulated by Schrödinger in late 1925 to describe the de Broglie waves of the electron within the hydrogen atom, as discussed by Kragh [6-13]. Due to its failure to account for the electron's spin, this equation led to conjectures that were inconsistent with empirical observations, resulting in its rejection by Pauli. Subsequently, the Klein–Gordon equation reemerged in seven separate papers, utilized to characterize relativistic massive particles lacking spin. However, during the initial stages of quantum mechanics, significant challenges arose regarding the interpretation of its role in relation to probability density [14-20]. It is now understood that these obstacles are resolved within the framework of quantum field theory, as noted by Kragh. The Klein-Gordon equation plays a crucial role in various fields, including mathematical physics, plasma physics, relativistic physics, nonlinear optics, and applied physical sciences [20-24].

This research delves into the realm of inhomogeneous nonlinear Klein-Gordon partial differential equations. Numerous numerical strategies have emerged to address such equations, such as the Homotopy analysis method, Variational iteration method, Homotopy perturbation method, and Adomian decomposition method, among others. Within this study, we present two novel numerical approaches tailored specifically for addressing linear or nonlinear Klein-Gordon equations. Firstly, we introduce the Zafar projected differential transform method (ZPDTM), which seamlessly integrates the Zafar transform method with the projected differential transform method. Secondly, we propose the Laplace projected differential transform method (LPDTM), which combines techniques from the Laplace transform method and the projected differential transform method.

The solutions derived through the Zafar Projected Differential Transform Method (ZPDTM) and the Laplace Projected Differential Transform Method (LPDTM) exhibit a wider range of applicability compared to those obtained using Variational Iteration Method (VIM), Adomian Decomposition Method (ADM), and Homotopy Perturbation Transform Method (HPTM) approaches. These methodologies demonstrate reliability and user-friendliness, highlighting their effectiveness and efficiency in obtaining precise and approximate solutions for nonlinear differential equations encountered in various scientific and engineering domains. Notably, ZPDTM and LPDTM significantly reduce computational burdens relative to traditional methods while maintaining high numerical precision. This reduction in computational overhead enhances the overall efficacy of these approaches. In conclusion, ZPDTM and LPDTM represent significant advancements in numerical techniques, with the potential for extensive utilization across diverse scientific and engineering fields. In this study, we present two innovative numerical methodologies tailored for resolving linear or nonlinear Klein-Gordon equations. Firstly, we introduce the Zafar Projected Differential Transform Method (ZPDTM), which seamlessly combines the Zafar transform method with the projected differential transform method. Secondly, we present the Laplace Projected Differential Transform Method (LPDTM), which integrates robust techniques from the Laplace transform method and the projected differential transform method.

Unlike ADM, HPM, and VIM, which often require iterative correction functions, nonlinear term decompositions, or construction of auxiliary functions, the proposed ZPDTM and LPDTM operate on a direct transformation framework. This framework minimizes symbolic computation and iterative procedures, leading to a reduction in computational overhead. Conceptually, the integration of Zafar and Laplace transforms within the projected differential transform framework allows for better handling of initial conditions and inhomogeneous terms, which is a common limitation in traditional methods. Furthermore, the recursive structure derived from the projected differential approach ensures a systematic and stable computation of solution terms without linearization or perturbation assumptions.

In summary, ZPDTM and LPDTM signify considerable progress in numerical methodologies, offering potential extensive adoption across diverse scientific and engineering fields. Table 1 shows some certain characteristics of Zafar and Laplace transformations.

Table 1: Certain characteristics of Zafar and Laplace transformations

$g(\tau)$	$Z\{g(\tau)\}$	$L\{g(\tau)\}$
1	1	1
τ	$\frac{v}{s}$	$\frac{1}{s}$
$\tau^n, n=1,2,3,4,\ldots$	$v!\frac{v^n}{s^n}$	$\frac{n!}{s^{n+1}}$
$\sin(\alpha\tau)$	$\frac{\alpha vs}{s^2+\alpha^2v^2}$	$\frac{\alpha}{s^2+\alpha^2}$
$e^{\alpha\tau}$	$\frac{s}{s-\alpha v}$	$\frac{1}{s-\alpha}$

MATERIALS

Overview of the Zafar Projected Differential Transform Method (Zpdtm)

We present the Zafar Projected Differential Transform Method (ZPDTM) using a non-homogeneous Klein-Gordon equation as a representative example:

$$v_{\tau\tau}(\varsigma,\tau)-u_{\varsigma\varsigma}(\varsigma,\tau)+k(v)=l(\varsigma,\tau),\ \varsigma>0,\tau>0 \tag{1}$$

Subject to the initial conditions:

$$v(\varsigma,0)=j_1(\varsigma), \tag{2}$$

The method begins by applying the Zafar transform with respect to τ, transforming the original equation into:

$$\frac{s^2}{v^2}Z\{v(\varsigma,\tau)\}-\frac{s^2}{v^2}v(\varsigma,0)-\frac{s}{v}v_{\varsigma}(\tau,0)=Z[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)] \tag{3}$$

Simplifying yields:

$$Z\{v(\varsigma,\tau)\}=v(\varsigma,0)+\frac{v}{s}v_{\varsigma}(\varsigma,0)+\frac{v^2}{s^2}Z[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)] \tag{4}$$

Substituting the initial conditions into equation (4), we obtain:

$$Z\{v(\varsigma,\tau)\}=j_1(\tau)+\frac{v}{s}j_2(\tau)+\frac{v^2}{s^2}Z[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)] \tag{5}$$

Applying the inverse Zafar transform, we recover the time-domain expression:

$$v(\varsigma,\tau)=j_1(\varsigma)+j_2(\varsigma)\tau+Z^{-1}\left\{\frac{v^2}{s^2}Z[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)]\right\} \tag{6}$$

The functions $j_1(\varsigma)$ and $j_2(\varsigma)$, represent the initial conditions. To derive the full series solution, we apply the Projected Differential Transform Method (PDTM) to equation (6):

$$\sum_{n=0}^{\infty}v_{n+1}(\varsigma,\tau)=Z^{-1}\left\{\frac{v^2}{s^2}Z[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)]\right\} \tag{7}$$

This leads to the recursive formulation:

$$v_{n+1}(\varsigma,\tau)=Z^{-1}\left\{\frac{v^2}{s^2}Z[A_n+B_n-C_n]\right\}; \tag{8}$$

Here, A_k, B_k and C_k are the projected differential transform of $l(\varsigma,\tau)$, $v_{\varsigma\varsigma}(\varsigma,\tau)$ and $k(v(\varsigma,\tau))$ respectively.

At $n=0$:

$$v_1(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z[A_0-B_0-C_0]\right] \tag{9}$$

At $n=1$:

$$v_2(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z[A_1-B_1-C_1]\right] \tag{10}$$

At $n=2$:

$$v_3(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z[A_2-B_2-C_2]\right] \tag{11}$$

Similarly, we can get the other series terms for $k=r$, which gives us the recursive relation as:

$$\left.\begin{aligned} v_0(\varsigma,\tau)&=j_1(\varsigma)+j_2(\varsigma)\tau+Z^{-1}\left\{\frac{v^2}{s^2}Z[l(\varsigma,\tau)+v_{xx}(\varsigma,\tau)-k(v)]\right\} \\ v_{r+1}(x,t)&=Z^{-1}\left[\frac{v^2}{s^2}Z[A_r-B_r-C_r]\right] \end{aligned}\right\} \tag{12}$$

Continuing this process yields the general series solution:

$$v(\varsigma,\tau)=\sum_{n=0}^{\infty}v_n(\varsigma,\tau) \tag{13}$$

Overview of the Laplace Projected Differential Transform Method (LPDTM)

We now outline the Laplace Projected Differential Transform Method (LPDTM), again illustrated using a non-homogeneous Klein-Gordon equation:

$$v_{\varsigma\varsigma}(\varsigma,\tau)-v_{\tau\tau}(\varsigma,\tau)+k(v)=l(\varsigma,\tau) \qquad \varsigma>0,\tau>0$$

subject to the initial conditions:

$$v(\varsigma,0)=j_1(\varsigma),\ v_\tau(\varsigma,0)=j_2(\varsigma)$$

The Laplace transform is applied with respect to τ, resulting in:

$$s^2L\{v(\varsigma,\tau)\}-sv(\varsigma,0)-v_\tau(\varsigma,0)=L[l(\varsigma,\tau)+u_{\tau\tau}(\varsigma,\tau)-k(v)] \tag{14}$$

Further simplification of Equation (14) yields:

$$L\{v(\varsigma,\tau)\}=\frac{1}{s}v(\varsigma,0)+\frac{1}{s^2}v_\tau(\varsigma,0)+\frac{1}{s^2}L[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)] \tag{15}$$

Substituting the initial conditions yields:

$$L\{v(\varsigma,\tau)\}=\frac{1}{s}j_1(\varsigma)+\frac{1}{s^2}j_2(\varsigma)+\frac{1}{s^2}L[l(\varsigma,\tau)+v_{\tau\tau}(\varsigma,\tau)-k(v)] \tag{16}$$

Taking the inverse Laplace transform gives:

$$L^{-1}\{L\{v(\varsigma,\tau)\}\}=L^{-1}\left\{\frac{1}{s}j_1(\varsigma)\right\}+L^{-1}\left\{\frac{1}{s^2}j_2(\varsigma)\right\}+L^{-1}\left\{\frac{1}{s^2}L\left[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)\right]\right\} \tag{17}$$

Then:

$$v(\varsigma,\tau)=j_1(\varsigma)+j_2(\varsigma)\tau+L^{-1}\left\{\frac{1}{s^2}L\left[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)\right]\right\} \tag{18}$$

The first series, denoted as $j_1(\varsigma)$ and $j_2(\varsigma)$, represent the prescribed initial conditions.

To obtain the series expansion of equation (18), we utilize the general form and apply the PDTM as demonstrated below:

$$\sum_{n=0}^{\infty}v_{n+1}(\varsigma,\tau)=L^{-1}\left\{\frac{1}{s^2}L\left[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)\right]\right\} \tag{19}$$

We then apply the PDTM to derive a series representation:

$$v_{n+1}(\varsigma,\tau)=L^{-1}\left\{\frac{1}{s^2}L\left[A_n+B_n-C_n\right]\right\}; \tag{20}$$

As in the ZPDTM approach, A_k, B_k and C_k are the projected differential transform of $l(\varsigma,\tau)$, $v_{\varsigma\varsigma}(\varsigma,\tau)$ and $k(v(\varsigma,\tau))$ respectively.

Similarly, we can get the other series terms for $k=r$, which gives us the recursive relation given below;

$$\left.\begin{array}{l} v_0(\varsigma,\tau)=j_1(\varsigma)+j_2(\varsigma)t+L^{-1}\left\{\frac{1}{s^2}L\left[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)\right]\right\} \\ v_{r+1}(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L\left[A_r-B_r-C_r\right]\right] \end{array}\right\} \tag{21}$$

With this we have the general approximate solution for Laplace Projected Differential Transform Method to be:

$$v(\varsigma,\tau)=\sum_{n=0}^{\infty}v_n(\varsigma,\tau) \tag{22}$$

METHODS

Case 1: The Homogeneous Linear Klein-Gordon Equation

Consider the homogeneous linear Klein-Gordon equation:

$$v_{\tau\tau}(\varsigma,\tau)=v_{\varsigma\varsigma}(\varsigma,\tau)-v(\varsigma,\tau) \tag{23}$$

Subject to initial conditions:

$$v(\varsigma,0)=0,\ v_\tau(\varsigma,0)=\varsigma \tag{24}$$

Method 1: Zafar Projected Differential Transform Method

$$\left.\begin{array}{l} v_0(\varsigma,\tau)=j_1(\varsigma)+j_2(\varsigma)\tau+Z^{-1}\left\{\frac{v^2}{s^2}Z\left[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)\right]\right\} \\ v_{r+1}(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[A_r-B_r-C_r\right]\right] \end{array}\right\} \tag{25}$$

Applying the initial conditions to equation (25) results in:

$$\left.\begin{aligned} v_0(\varsigma,\tau) &= 0+\varsigma\tau+Z^{-1}\left\{\frac{v^2}{s^2}Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-v(\varsigma,\tau)\right]\right\} \\ v_{r+1}(\varsigma,\tau) &= Z^{-1}\left[\frac{v^2}{s^2}Z\left[A_r-B_r\right]\right] \end{aligned}\right\} \tag{26}$$

From this, we obtain: $v_0(\varsigma,\tau)=0$, $v_1(\varsigma,\tau)=\varsigma\tau$,

The recursive solution is obtained using the following relation:

$$v_{r+1}(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[A_r-B_r\right]\right] \tag{27}$$

Where, $A_r=\left(v_r(\varsigma,\tau)\right)_{\varsigma\varsigma}$, and $B_r=\left(v_r(\varsigma,\tau)\right)$

At $r=1$, we compute:

$$v_2(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[A_1-B_1\right]\right]$$

$A_1=\left(v_1(\varsigma,\tau)\right)_{\varsigma\varsigma}=\left(\varsigma\tau\right)_{\varsigma\varsigma}=0$ and $B_1=\left(v_1(\varsigma,\tau)\right)=\varsigma\tau$

and

$$v_2(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[-\varsigma\tau\right]\right]=Z^{-1}\left[-\varsigma\frac{v^2}{s^2}.\frac{v}{s}\right]=-\frac{\varsigma}{3!}.Z^{-1}\left[\frac{3!v^3}{s^3}\right]=-\frac{\varsigma}{3!}\tau^3 \tag{28}$$

At $r=2$, we have:

$$v_3(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[A_2-B_2\right]\right]$$

$A_2=\left(v_2(\varsigma,\tau)\right)_{\varsigma\varsigma}=\left(-\frac{\varsigma}{3!}\tau^3\right)_{\varsigma\varsigma}=0$ and $B_1=\left(-\frac{\varsigma}{3!}\tau^3\right)=-\frac{\varsigma}{3!}\tau^3$

and

$$v_3(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[\frac{\varsigma}{3!}\tau^3\right]\right]=Z^{-1}\left[\frac{\varsigma}{3!}\frac{v^2}{s^2}.3!\frac{v^3}{s^3}\right]=\frac{\varsigma}{5!}.Z^{-1}\left[\frac{5!v^5}{s^5}\right]=\frac{\varsigma}{5!}\tau^5 \tag{29}$$

At $r=3$, we obtain:

$$v_4(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[A_3-B_3\right]\right]$$

$A_3=\left(v_3(\varsigma,\tau)\right)_{\varsigma\varsigma}=\left(\frac{\varsigma}{5!}\tau^5\right)_{\varsigma\varsigma}=0$ and $B_1=\left(\frac{\varsigma}{5!}\tau^5\right)=\frac{\varsigma}{5!}\tau^5$

and

$$v_4(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[-\frac{\varsigma}{5!}\tau^5\right]\right]=Z^{-1}\left[-\frac{\varsigma}{5!}\frac{v^2}{s^2}.5!\frac{v^5}{s^5}\right]=-\frac{\varsigma}{7!}.Z^{-1}\left[\frac{7!v^7}{s^7}\right]=-\frac{\varsigma}{7!}\tau^7 \tag{30}$$

At $r=4$, we find:

$$v_5(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[A_4-B_4\right]\right]$$

$A_4 = \left(v_4(\varsigma,\tau)\right)_{\varsigma\varsigma} = \left(-\frac{\varsigma}{7!}\tau^7\right)_{\varsigma\varsigma} = 0$ and $B_1 = \left(-\frac{\varsigma}{7!}\tau^7\right) = -\frac{\varsigma}{7!}\tau^7$

and

$$v_5(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2}Z\left[\frac{\varsigma}{7!}\tau^7\right]\right] = Z^{-1}\left[\frac{\varsigma}{7!}\frac{v^2}{s^2}.7!\frac{v^7}{s^7}\right] = \frac{\varsigma}{9!}.Z^{-1}\left[\frac{9!v^9}{s^9}\right] = \frac{\varsigma}{9!}\tau^9 \tag{31}$$

Continuing in this manner, the approximate solution obtained via ZPDTM is given by:

$$v(\varsigma,\tau) = \varsigma\left(\tau - \frac{1}{3!}\tau^3 + \frac{1}{5!}\tau^5 - \frac{1}{7!}\tau^7 + \frac{1}{9!}\tau^9 - \ldots\right) = \varsigma\sin\tau \tag{32}$$

Method 2: Using Laplace Projected Differential Transform Method

$$\left.\begin{aligned} v_0(\varsigma,\tau) &= j_1(\varsigma) + j_2(\varsigma)\tau + L^{-1}\left\{\frac{1}{s^2}L\left[l(\varsigma,\tau) + v_{\varsigma\varsigma}(\varsigma,\tau) - k(v)\right]\right\} \\ v_{r+1}(\varsigma,\tau) &= L^{-1}\left[\frac{1}{s^2}L\left[A_r - B_r - C_r\right]\right] \end{aligned}\right\} \tag{33}$$

Applying the Laplace transforms to equation (23) and incorporating the initial conditions yields:

$$\left.\begin{aligned} v_0(\varsigma,\tau) &= 0 + \varsigma\tau + L^{-1}\left\{\frac{1}{s^2}L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - v(\varsigma,\tau)\right]\right\} \\ v_{r+1}(\varsigma,\tau) &= L^{-1}\left[\frac{1}{s^2}L\left[A_r - B_r\right]\right] \end{aligned}\right\} \tag{34}$$

Using the initial conditions:

$$v_0(\varsigma,\tau) = 0,\ v_1(\varsigma,\tau) = \varsigma\tau$$

we derive the recurrence relation:

$$v_{r+1}(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}L\left[A_r - B_r\right]\right]$$

Where, $A_r = \left(v_r(\varsigma,\tau)\right)_{\varsigma\varsigma}$, and $B_r = \left(v_r(\varsigma,\tau)\right)$

At $r = 1$, we compute:

$$v_2(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}Z\left[A_1 - B_1\right]\right]$$

$A_1 = \left(v_1(\varsigma,\tau)\right)_{\varsigma\varsigma} = (\varsigma\tau)_{\varsigma\varsigma} = 0$ and $B_1 = \left(v_1(\varsigma,\tau)\right) = \varsigma\tau$

$$v_2(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}L\left[-\varsigma\tau\right]\right] = Z^{-1}\left[-\varsigma\frac{1}{s^2}.\frac{1}{s^2}\right] = -\frac{\varsigma}{3!}.Z^{-1}\left[3!.\frac{1}{s^4}\right] = -\frac{\varsigma}{3!}\tau^3 \tag{35}$$

At $r = 3$, we have:

$$v_3(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}Z\left[A_2 - B_2\right]\right]$$

$$A_2=\left(v_2(\varsigma,\tau)\right)_{\varsigma\varsigma}=\left(-\frac{\varsigma}{3!}\tau^3\right)_{\varsigma\varsigma}=0 \text{ and } B_2=\left(-\frac{\varsigma}{3!}\tau^3\right)=-\frac{\varsigma}{3!}\tau^3$$

$$v_3(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L\left[\frac{\varsigma}{3!}\tau^3\right]\right]=L^{-1}\left[\frac{\varsigma}{3!}\frac{1}{s^2}.3!\frac{1}{s^4}\right]=\frac{\varsigma}{5!}.L^{-1}\left[5!.\frac{1}{s^6}\right]=\frac{\varsigma}{5!}\tau^5 \tag{36}$$

At $r=3$, we obtain:

$$v_4(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L\left[A_3-B_3\right]\right]$$

$$A_3=\left(v_3(\varsigma,\tau)\right)_{xx}=\left(\frac{\varsigma}{5!}\tau^5\right)_{\varsigma\varsigma}=0 \text{ and } B_3=\left(\frac{\varsigma}{5!}\tau^5\right)=\frac{\varsigma}{5!}\tau^5$$

$$v_4(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L\left[-\frac{\varsigma}{5!}\tau^5\right]\right]=L^{-1}\left[-\frac{\varsigma}{5!}\frac{1}{s^2}.5!\frac{1}{s^6}\right]=-\frac{\varsigma}{7!}.Z^{-1}\left[7!\frac{1}{s^8}\right]=-\frac{\varsigma}{7!}\tau^7 \tag{37}$$

At $r=4$, we compute:

$$v_5(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L\left[A_4-B_4\right]\right]$$

$$A_4=\left(v_4(\varsigma,\tau)\right)_{\varsigma\varsigma}=\left(-\frac{\varsigma}{7!}\tau^7\right)_{\varsigma\varsigma}=0 \text{ and } B_4=\left(-\frac{\varsigma}{7!}\tau^7\right)=-\frac{\varsigma}{7!}\tau^7$$

$$v_5(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L\left[\frac{\varsigma}{7!}\tau^7\right]\right]=L^{-1}\left[\frac{\varsigma}{7!}\frac{1}{s^2}.7!\frac{1}{s^8}\right]=\frac{\varsigma}{9!}.Z^{-1}\left[9!\frac{1}{s^{10}}\right]=\frac{\varsigma}{9!}\tau^9 \tag{38}$$

Continuing similarly, the LPDTM solution with the given initial conditions is expressed as:

$$v(\varsigma,\tau)=\varsigma\left(\tau-\frac{1}{3!}\tau^3+\frac{1}{5!}\tau^5-\frac{1}{7!}\tau^7+\frac{1}{9!}\tau^9-\ldots\right)=\varsigma\sin\tau \tag{39}$$

Figure 1 presents the exact solution profile for the homogeneous linear Klein-Gordon equation (case 1) using ZPDTM and LPDTM. Both methods yield identical results, confirming consistency and accuracy.

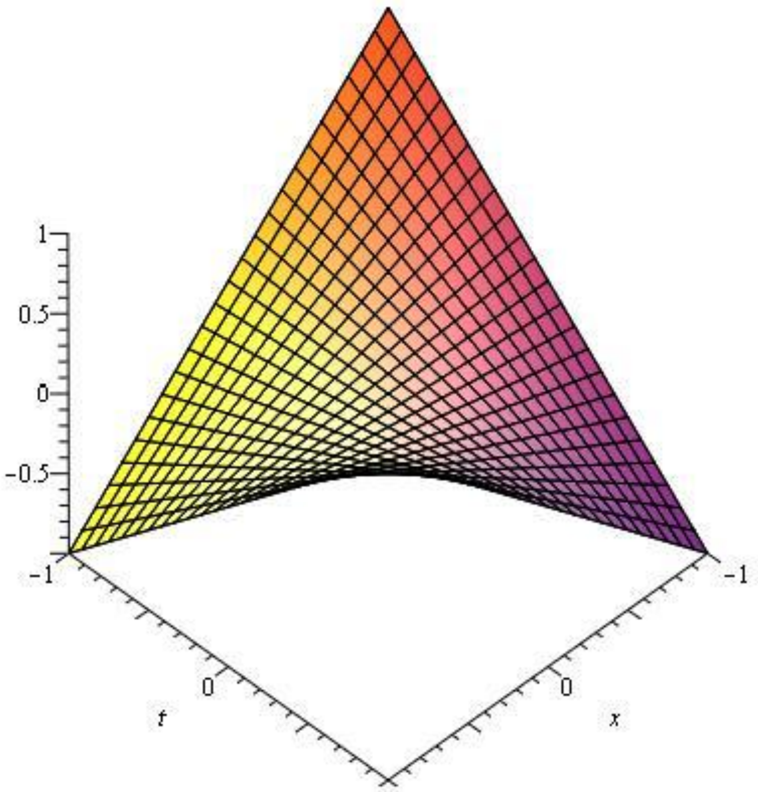

Figure 1: Exact solution for the homogeneous linear Klein-Gordon equation (case 1) using ZPDTM and LPDTM. Both methods yield identical results, confirming their computational accuracy and consistency.

Case 2: The Non-Homogeneous Linear Klein-Gordon Equation

Consider the non-homogeneous linear Klein-Gordon equation:

$$v_{\varsigma\varsigma}(\varsigma,\tau)=v_{\varsigma\varsigma}(\varsigma,\tau)-v(\varsigma,\tau)+2\sin(\varsigma) \tag{40}$$

subject to the initial conditions:

$$v(\varsigma,0)=\sin(\varsigma),\ v_{\tau}(\varsigma,0)=1 \tag{41}$$

Method 1: Zafar Projected Differential Transform Method

$$\left.\begin{aligned}
&v_0(\varsigma,\tau)=j_1(\varsigma)+j_2(\varsigma)\tau+Z^{-1}\left\{\frac{v^2}{s^2}Z\left[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)\right]\right\}\\
&v_{r+1}(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[A_r-B_r-C_r\right]\right]
\end{aligned}\right\} \tag{42}$$

Applying the Zafar transform to equation (40) and incorporating the initial conditions results in:

$$\left.\begin{aligned}
&v_0(\varsigma,\tau)=\sin(\varsigma)+\tau+Z^{-1}\left\{\frac{v^2}{s^2}Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-v(\varsigma,\tau)+2\sin(\varsigma)\right]\right\}\\
&v_{r+1}(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[A_r-B_r\right]\right]
\end{aligned}\right\} \tag{43}$$

From this, we have:

$$v_0(\varsigma,\tau)=\sin(\varsigma),\ v_1(\varsigma,\tau)=\tau$$

The recursive relation is given by:

$$v_{r+1}(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z[A_r-B_r+2\sin(\varsigma)]\right]$$

Where, $A_r=(v_r(\varsigma,\tau))_{\varsigma\varsigma}$, and $B_r=(v_r(\varsigma,\tau))$

At $r=1$, we compute:

$$v_2(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z[A_1-B_1+2\sin(\varsigma)]\right]$$

$A_1=(v_1(\varsigma,\tau))_{\varsigma\varsigma}=(\tau)_{\varsigma\varsigma}=0$ and $B_1=(v_1(\tau))=\tau$

$$v_2(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z[-\tau+2\sin(\varsigma)]\right]=Z^{-1}\left[\frac{v^2}{s^2}\left(-\frac{v}{s}\right)\right]=.Z^{-1}\left[-\frac{1}{3!}.3!.\frac{v^3}{s^3}\right]=-\frac{1}{3!}\tau^3 \tag{44}$$

At $r=2$, we have:

$$v_3(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z[A_2-B_2+2\sin(\varsigma)]\right]$$

$A_2=(v_2(\varsigma,\tau))_{\varsigma\varsigma}=\left(-\frac{1}{3!}\tau^3\right)_{\varsigma\varsigma}=0$ and $B_2=\left(-\frac{1}{3!}\tau^3\right)=-\frac{1}{3!}\tau^3$

Putting above equation and simplified, we have,

$$v_3(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[\frac{1}{3!}\tau^3\right]\right]=Z^{-1}\left(\frac{1}{5!}5!\frac{v^5}{s^5}\right)=\frac{1}{5!}\tau^5 \tag{45}$$

At $r=3$, we obtain:

$$v_4(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z[A_3-B_3+2\sin(\varsigma)]\right]$$

$A_3=\left(\frac{1}{5!}\tau^5\right)_{\varsigma\varsigma}=0$ and $B_3=\left(\frac{1}{5!}\tau^5\right)=\frac{1}{5!}\tau^5$

$$v_4(\varsigma,\tau)=Z^{-1}\left[\frac{v^2}{s^2}Z\left[-\frac{1}{5!}\tau^5\right]\right]=Z^{-1}\left[-\frac{1}{7!}7!\frac{v^7}{s^7}\right]=-\frac{1}{7!}\tau^7 \tag{46}$$

Continuing this pattern, the approximate ZPDTM solution is given by:

$$v(\varsigma,\tau)=\sin\varsigma+\left(\tau-\frac{1}{3!}\tau^3+\frac{1}{5!}\tau^5-\frac{1}{7!}\tau^7+\ldots\right)=\sin\varsigma+\sin\tau \tag{47}$$

Method 2: Using Laplace Projected Differential Transform Method

$$\left.\begin{aligned}&v_0(\varsigma,\tau)=j_1(\varsigma)+j_2(\varsigma)\tau+L^{-1}\left\{\frac{1}{s^2}L[l(\varsigma,\tau)+v_{\varsigma\varsigma}(\varsigma,\tau)-k(v)]\right\}\\&v_{r+1}(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L[A_r-B_r-C_r]\right]\end{aligned}\right\} \tag{48}$$

Substituting the initial conditions into equation (48), we have:

$$\left.\begin{aligned} v_0(\varsigma,\tau) &= \sin(\varsigma) + t + L^{-1}\left\{\frac{1}{s^2}L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - v(\varsigma,\tau) + 2\sin(\varsigma)\right]\right\} \\ v_{r+1}(\varsigma,\tau) &= L^{-1}\left[\frac{1}{s^2}L\left[A_r - B_r\right]\right] \end{aligned}\right\} \tag{49}$$

Thus:

$$v_0(\varsigma,\tau) = \sin(\varsigma),\ v_1(\varsigma,\tau) = \tau$$

The recursive relation for the LPDTM method is:

$$v_{r+1}(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}L\left[A_r - B_r + 2\sin(\varsigma)\right]\right]$$

Where, $A_r = (v_r(\varsigma,\tau))_{\varsigma\varsigma}$, and $B_r = (v_r(\varsigma,\tau))$

At $r = 1$:

$$v_2(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}L\left[A_1 - B_1 + 2\sin(\varsigma)\right]\right]$$

$A_1 = (v_1(\varsigma,\tau))_{\varsigma\varsigma} = (\tau)_{\varsigma\varsigma} = 0$ and $B_1 = (v_1(\tau)) = \tau$

$$v_2(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}L\left[-\tau + 2\sin(\varsigma)\right]\right] = L^{-1}\left[\frac{1}{s^2}\left(-\frac{1}{s^2}\right)\right] = .L^{-1}\left[-\frac{1}{3!}.3!.\frac{1}{s^4}\right] = -\frac{1}{3!}\tau^3 \tag{50}$$

At $r = 2$:

$$v_3(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}L\left[A_2 - B_2 + 2\sin(\varsigma)\right]\right]$$

$A_2 = (v_2(\varsigma,\tau))_{\varsigma\varsigma} = \left(-\frac{1}{3!}\tau^3\right)_{\varsigma\varsigma} = 0$ and $B_2 = \left(-\frac{1}{3!}\tau^3\right) = -\frac{1}{3!}\tau^3$

Putting above equation and simplified, we have,

$$v_3(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}L\left[\frac{1}{3!}\tau^3\right]\right] = L^{-1}\left(\frac{1}{5!}5!\frac{1}{s^6}\right) = \frac{1}{5!}\tau^5 \tag{51}$$

At $r = 3$:

$$v_4(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}L\left[A_3 - B_3 + 2\sin(\varsigma)\right]\right]$$

$A_3 = \left(\frac{1}{5!}\tau^5\right)_{\varsigma\varsigma} = 0$ and $B_3 = \left(\frac{1}{5!}\tau^5\right) = \frac{1}{5!}\tau^5$

$$v_4(\varsigma,\tau) = L^{-1}\left[\frac{1}{s^2}L\left[-\frac{1}{5!}\tau^5\right]\right] = L^{-1}\left[-\frac{1}{7!}7!\frac{1}{s^8}\right] = -\frac{1}{7!}\tau^7 \tag{52}$$

Following this procedure, the LPDTM approximate solution is:

$$v(\varsigma,\tau) = \sin\varsigma + \left(\tau - \frac{1}{3!}\tau^3 + \frac{1}{5!}\tau^5 - \frac{1}{7!}\tau^7 + \ldots\right) = \sin\varsigma + \sin\tau \tag{53}$$

Figure 2 presents Comparison of ZPDTM and LPDTM solutions for the non-homogeneous linear Klein-Gordon equation (case 2). The overlapping curves illustrate both methods' ability to recover the exact solution without error or divergence.

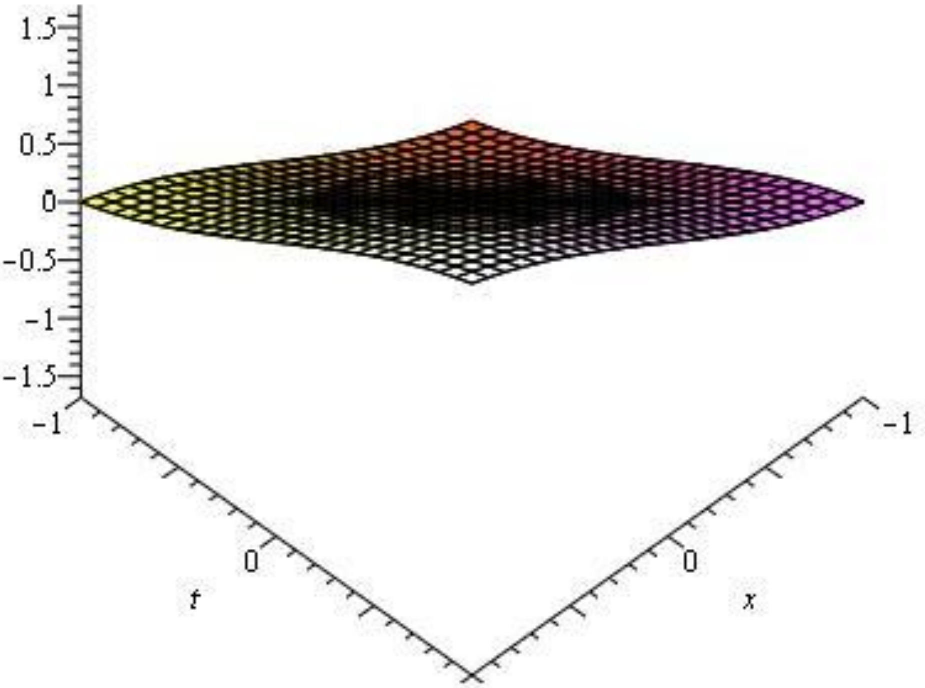

Figure 2: ZPDTM and LPDTM solution curves for the non-homogeneous linear Klein-Gordon equation (case 2). The figure verifies that both methods accurately capture the effect of the forcing term, maintaining exactness and consistency without requiring iterative correction or decomposition.

Case 3: The Non-Homogeneous Non-Linear Klein-Gordon Equation

Consider the non-homogeneous nonlinear Klein-Gordon equation:

$$v_{\tau\tau}(\varsigma,\tau)=\varsigma^2\tau^2+v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2 \tag{54}$$

subject to the initial conditions:

$$v(\varsigma,0)=0\,,\ v_\tau(\varsigma,0)=\varsigma \tag{55}$$

Method 1: Using Zafar Projected Differential Transform Method

We begin by applying the Zafar transformation to equation (54) with respect to τ, resulting in:

$$\frac{s^2}{v^2}Z\{v(\varsigma,\tau)\}-\frac{s^2}{v^2}v(\varsigma,0)-\frac{s}{v}v_\tau(\varsigma,0)=\varsigma^2.2!.\frac{v^2}{s^2}+Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right] \tag{56}$$

Simplifying equation (56), we obtain:

$$Z\{v(\varsigma,\tau)\}=v(\varsigma,0)+\frac{v}{s}v_\tau(\tau,0)+2\varsigma^2\frac{v^4}{s^4}+\frac{v^2}{s^2}Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right] \tag{57}$$

Substituting the initial condition into the equation yields:

$$Z\{v(\varsigma,\tau)\}=0+\frac{v}{s}\varsigma+2\varsigma^2\frac{v^4}{s^4}+\frac{v^2}{s^2}Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right] \tag{58}$$

Taking the inverse Zafar transform, we get:

$$v(\varsigma,\tau)=0+\varsigma\tau+2\varsigma^2\frac{\tau^2}{4!}+Z^{-1}\left\{\frac{v^2}{s^2}Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right]\right\} \tag{59}$$

To derive the series solution, we apply the Projected Differential Transform Method (PDTM):

$$\sum_{r=0}^{\infty} v_{r+1}(\varsigma,\tau) = Z^{-1}\left\{\frac{v^2}{s^2} Z\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right]\right\} \tag{60}$$

By applying the PDTM, we obtain:

$$v_{r+1}(\varsigma,\tau) = Z^{-1}\left\{\frac{v^2}{s^2} Z[A_r - B_r]\right\}; \tag{61}$$

Here, A_k and B_k represent the projected differential transform of $v_{\varsigma\varsigma}(\varsigma,\tau)$ and $(v(\varsigma,\tau))^2$ respectively.

At $r = 1$, we compute:

$$v_2(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[A_1 - B_1]\right]$$

$$A_1 = [v_1(\varsigma,\tau)]_{\varsigma\varsigma} = (\varsigma\tau)_{\varsigma\varsigma} = 0;$$

$$B_1 = \sum_{m=0}^{r=1} v_m(\varsigma,\tau) u_{r-m}(\varsigma,\tau) = v_0(\varsigma,\tau)v_1(\varsigma,\tau) + v_1(\varsigma,\tau)v_0(\varsigma,\tau) = 0.\varsigma\tau + \varsigma\tau.0 = 0$$

$$v_2(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[0-0]\right] = Z^{-1}[0] = 0 \tag{62}$$

At $r = 2$, we have:

$$v_3(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[A_2 - B_2]\right]$$

$$A_2 = [v_2(\varsigma,\tau)]_{\varsigma\varsigma} = (0)_{\varsigma\varsigma} = 0;$$

$$B_2 = \sum_{m=0}^{r=2} v_m(\varsigma,\tau) v_{r-m}(\varsigma,\tau) = 0$$

$$v_3(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[0-0]\right] = Z^{-1}[0] = 0 \tag{63}$$

Continuing in this manner, the approximate solution obtained via ZPDTM is:

$$v(\varsigma,\tau) = \varsigma\tau + 0 + 0 + 0 + 0 + \ldots = \varsigma\tau \tag{64}$$

Method 2: Laplace Projected Differential Transform Method

Applying the Laplace transformation to equation (54) with respect to τ, we obtain:

$$s^2 L\{v(\varsigma,\tau)\} - s^2 v(\varsigma,0) - s v_\tau(\varsigma,0) = 2\varsigma^2 \frac{1}{s^3} + L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right] \tag{65}$$

Simplifying equation (65), we get:

$$L\{v(\varsigma,\tau)\} = v(\varsigma,0) + \frac{1}{s} v_\tau(\varsigma,0) + 2\varsigma^2 \frac{1}{s^5} + \frac{1}{s^2} L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right] \tag{66}$$

Incorporating the initial conditions leads to:

$$L\{v(\varsigma,\tau)\} = 0 + \frac{v}{s}\varsigma + 2\varsigma^2 \frac{1}{s^5} + \frac{1}{s^2} L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right] \tag{67}$$

Taking the inverse Laplace transform gives:

$$v(\varsigma,\tau)=0+\varsigma\tau+\varsigma^2\frac{\tau^4}{12}+L^{-1}\left\{\frac{v^2}{s^2}L\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right]\right\} \tag{68}$$

We then apply the PDTM to obtain the series form:

$$\sum_{r=0}^{\infty}v_{r+1}(\varsigma,\tau)=L^{-1}\left\{\frac{1}{s^2}L\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right]\right\} \tag{69}$$

By applying the PDTM, we find:

$$v_{r+1}(\varsigma,\tau)=L^{-1}\left\{\frac{1}{s^2}L[A_r-B_r]\right\}; \tag{70}$$

Where A_k and B_k denote the projected differential transform of $v_{\varsigma\varsigma}(\varsigma,\tau)$ and the nonlinear source term $(v(\varsigma,\tau))^2$ respectively.

At $r=1$:

$$v_2(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L[A_1-B_1]\right]$$

$$A_1=[v_1(\varsigma,\tau)]_{\varsigma\varsigma}=(\varsigma\tau)_{\varsigma\varsigma}=0;$$

$$B_1=\sum_{m=0}^{r=1}v_m(\varsigma,\tau)v_{r-m}(\varsigma,\tau)=v_0(\varsigma,\tau)v_1(\varsigma,\tau)+v_1(\varsigma,\tau)v_0(\varsigma,\tau)=0.\varsigma\tau+\varsigma\tau.0=0$$

$$v_2(\varsigma,\tau)=L^{-1}\left[\frac{v^2}{s^2}L[0-0]\right]=L^{-1}[0]=0 \tag{71}$$

At $r=2$:

$$v_3(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L[A_2-B_1]\right]$$

$$A_2=[v_2(\varsigma,\tau)]_{\varsigma\varsigma}=(0)_{\varsigma\varsigma}=0;$$

$$B_2=\sum_{m=0}^{r=2}v_m(\varsigma,\tau)v_{r-m}(\varsigma,\tau)=0$$

$$v_3(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L[0-0]\right]=L^{-1}[0]=0 \tag{72}$$

Continuing in this fashion, the LPDTM solution is expressed as:

$$v(\varsigma,\tau)=\varsigma\tau+0+0+0+0+\ldots=\varsigma\tau \tag{73}$$

Figure 3 illustrates the Solution curves for the non-homogeneous nonlinear Klein-Gordon equation (case 3) obtained via ZPDTM and LPDTM. The figure highlights the effectiveness of both approaches in accurately solving nonlinear models.

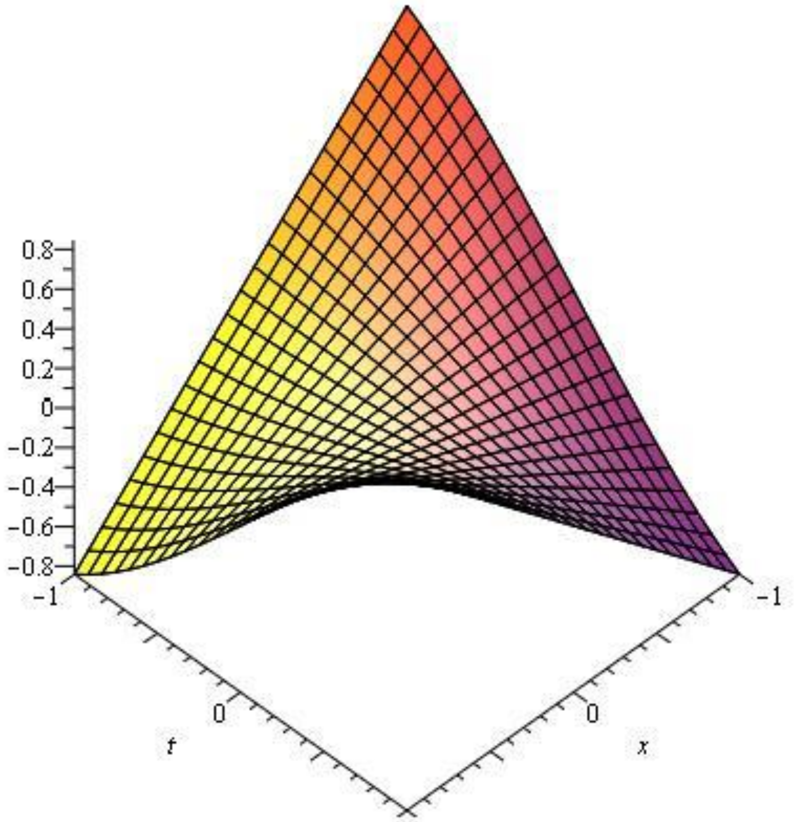

Figure 3: Graphical comparison of ZPDTM and LPDTM solutions for the non-homogeneous nonlinear Klein-Gordon equation (case 3). The matching curves highlight both methods' capacity to handle nonlinearities while producing exact or near-exact solutions efficiently.

Case 4: The Non-Homogeneous Non-Linear Klein-Gordon Equation

Consider the non-homogeneous nonlinear Klein-Gordon equation:

$$v_{\tau\tau}(\varsigma,\tau)=2\varsigma^2-2\tau^2+\varsigma^4\tau^4+v_{\varsigma\varsigma}(\varsigma,\tau)-(u(\varsigma,\tau))^2 \tag{74}$$

subject to the initial conditions:

$$v(\varsigma,0)=0,\ v_{\tau}(\varsigma,0)=0 \tag{75}$$

Method 1: Using Zafar Projected Differential Transform Method

We begin by applying the Zafar transform to equation (74) with respect to τ, resulting in:

$$\frac{s^2}{v^2}Z\{v(\varsigma,\tau)\}-\frac{s^2}{v^2}v(\varsigma,0)-\frac{s}{v}v_t(\varsigma,0)=2\varsigma^2-2.2!\frac{v^2}{s^2}+x^4.4!\frac{v^4}{s^4}+Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right] \tag{76}$$

Simplifying equation (76), we obtain:

$$Z\{u(x,t)\}=u(x,0)+\frac{u}{s}u_t(x,0)+x^2 2!\frac{u^2}{s^2}-2.2!+x^4.4!\frac{u^2}{s^2}+\frac{u^2}{s^2}Z\left[u_{xx}(x,t)-(u(x,t))^2\right] \tag{77}$$

Substituting the initial condition into equation (77) yields:

$$Z\{v(\varsigma,\tau)\}=\varsigma^2 2!\frac{v^2}{s^2}-2.2!\frac{v^4}{s^4}+x^4.4!\frac{v^6}{s^6}+\frac{v^2}{s^2}Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right] \tag{78}$$

Taking the inverse Zafar transform, we derive:

$$v(\varsigma,\tau)=\varsigma^2\tau^2-\frac{1}{6}\tau^4+\varsigma^4.\frac{1}{30}\tau^6+Z^{-1}\left\{\frac{v^2}{s^2}Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right]\right\} \tag{79}$$

To determine the series expansion, we apply the Projected Differential Transform Method (PDTM):

$$\sum_{r=0}^{\infty} v_{r+1}(\varsigma,\tau) = Z^{-1}\left\{\frac{v^2}{s^2} Z\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right]\right\} \tag{80}$$

Applying the PDTM gives:

$$v_{r+1}(\varsigma,\tau) = Z^{-1}\left\{\frac{1}{s^2} L[A_r - B_r]\right\}; \tag{81}$$

Here, A_k and B_k are the projected differential transform of $v_{\varsigma\varsigma}(\varsigma,\tau)$ and the nonlinear term $(v(\varsigma,\tau))^2$ respectively.

At $r = 1$:

$$v_2(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[A_1 - B_1]\right]$$

$$A_1 = [v_1(\varsigma,\tau)]_{\varsigma\varsigma} = (0)_{\varsigma\varsigma} = 0;$$

$$B_1 = \sum_{m=0}^{r=1} v_m(\varsigma,\tau) v_{r-m}(\varsigma,\tau) = v_0(\varsigma,\tau) v_1(\varsigma,\tau) + v_1(\varsigma,\tau) v_0(\varsigma,\tau) = 0.\varsigma\tau + \varsigma\tau.0 = 0$$

$$v_2(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[0-0]\right] = Z^{-1}[0] = 0 \tag{82}$$

At $r = 2$:

$$v_3(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[A_2 - B_2]\right]$$

$$A_2 = [v_2(\varsigma,\tau)]_{\varsigma\varsigma} = (0)_{\varsigma\varsigma} = 0;$$

$$B_2 = \sum_{m=0}^{r=2} v_m(\varsigma,\tau) v_{r-m}(\varsigma,\tau) = 0$$

$$v_3(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[0-0]\right] = Z^{-1}[0] = 0 \tag{83}$$

Continuing this process, the approximate solution using ZPDTM is given by:

$$v(\varsigma,\tau) = \varsigma^2 \tau^2 \tag{84}$$

Method 2: Using Laplace Projected Differential Transform Method

Applying the Laplace transformation to equation (74) with respect to τ, we obtain:

$$s^2 L\{v(\varsigma,\tau)\} - s^2 v(\varsigma,0) - s v_t(\varsigma,0) = \frac{2\varsigma^2}{s} - 2.2!\frac{1}{s^3} + \varsigma^4.4!\frac{1}{s^5} + L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right] \tag{85}$$

Simplifying equation (85), we arrive at:

$$L\{v(\varsigma,\tau)\} = v(\varsigma,0) + \frac{1}{s} v_\tau(\varsigma,0) + \varsigma^2 2!\frac{1}{s^3} - 2.2!\frac{1}{s^5} + \varsigma^4.4!\frac{1}{s^7} + \frac{1}{s^2} L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right] \tag{86}$$

Substituting the initial condition gives:

$$L\{v(\varsigma,\tau)\} = \varsigma^2 2!\frac{1}{s^3} - 2.2!\frac{1}{s^5} + \varsigma^4.4!\frac{1}{s^7} + \frac{1}{s^2} L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right] \tag{87}$$

Taking the inverse Laplace transform, we get:

$$L^{-1}\{L\{v(\varsigma,\tau)\}\}=\varsigma^{2}2!\frac{1}{s^{3}}-2.2!\frac{1}{s^{5}}+\varsigma^{4}.4!\frac{1}{s^{7}}+L^{-1}\left\{\frac{1}{s^{2}}L\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^{2}\right]\right\} \tag{88}$$

Then, the expression becomes:

$$v(\varsigma,\tau)=\varsigma^{2}\tau^{2}-\frac{1}{6}\tau^{4}+\varsigma^{4}.\frac{1}{30}\tau^{6}+L^{-1}\left\{\frac{1}{s^{2}}L\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^{2}\right]\right\} \tag{89}$$

Applying the PDTM to this equation results in:

$$\sum_{r=0}^{\infty}v_{r+1}(\varsigma,\tau)=L^{-1}\left\{\frac{1}{s^{2}}L\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^{2}\right]\right\} \tag{90}$$

Using the PDTM yields:

$$v_{r+1}(\varsigma,\tau)=L^{-1}\left\{\frac{1}{s^{2}}L[A_{r}-B_{r}]\right\}; \tag{91}$$

Where A_k and B_k are the projected differential transform of $v_{\varsigma\varsigma}(\varsigma,\tau)$ and the nonlinear source term $(v(\varsigma,\tau))^2$, respectively.

At $r=1$:

$$v_{2}(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^{2}}L[A_{1}-B_{1}]\right]$$

$$A_{1}=[v_{1}(\varsigma,\tau)]_{\varsigma\varsigma}=(0)_{\varsigma\varsigma}=0;$$

$$B_{1}=\sum_{m=0}^{r=1}v_{m}(\varsigma,\tau)v_{r-m}(\varsigma,\tau)=v_{0}(\varsigma,\tau)v_{1}(\varsigma,\tau)+v_{1}(\varsigma,\tau)v_{0}(\varsigma,\tau)=0.\varsigma\tau+\varsigma\tau.0=0$$

$$v_{2}(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^{2}}L[0-0]\right]=L^{-1}[0]=0 \tag{92}$$

At $r=2$:

$$v_{3}(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^{2}}L[A_{2}-B_{2}]\right]$$

$$A_{2}=[v_{2}(\varsigma,\tau)]_{\varsigma\varsigma}=(0)_{\varsigma\varsigma}=0;$$

$$B_{2}=\sum_{m=0}^{r=2}v_{m}(\varsigma,\tau)v_{r-m}(\varsigma,\tau)=0$$

$$v_{3}(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^{2}}L[0-0]\right]=L^{-1}[0]=0 \tag{93}$$

Proceeding similarly, the LPDTM approximate solution is given by:

$$v(\varsigma,\tau)=\varsigma^{2}\tau^{2} \tag{94}$$

Figure 4 provides computed solutions using ZPDTM and LPDTM for a second non-homogeneous nonlinear Klein-Gordon equation (case 4). The identical curves reinforce the validity and robustness of the methods.

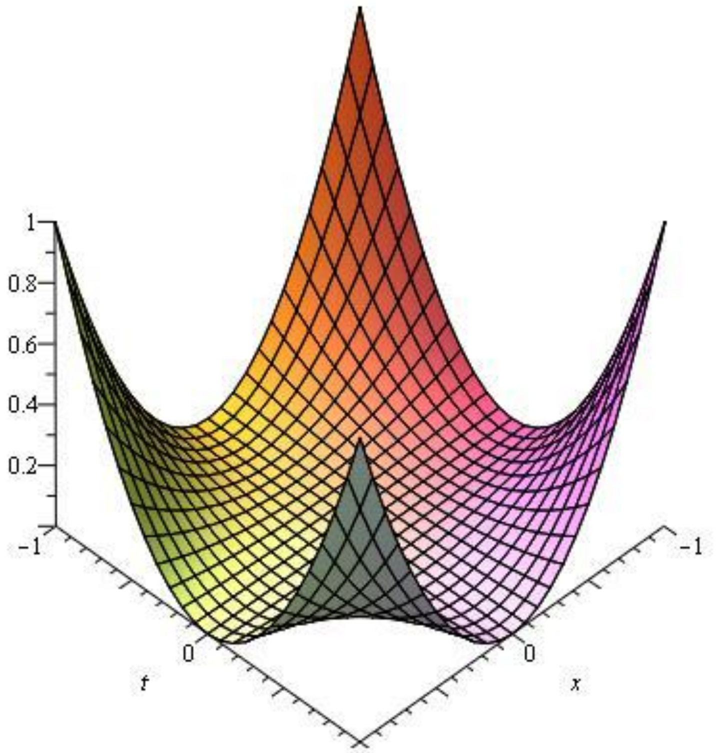

Figure 4: Solutions obtained using ZPDTM and LPDTM for a second nonlinear non-homogeneous Klein-Gordon (case 4). The agreement between both methods emphasizes their robustness in preserving solution behavior under more complex nonlinear interactions.

Case 5: The Non-Homogeneous Non-Linear Klein-Gordon Equation

Consider the non-homogeneous nonlinear Klein-Gordon equation:

$$v_{\tau\tau}(\varsigma,\tau)=6\varsigma\tau\left(\varsigma^2-\tau^2\right)+\varsigma^6\tau^6+v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2 \tag{95}$$

subject to the initial conditions:

$$v(\varsigma,0)=0,\ v_\tau(\varsigma,0)=0 \tag{96}$$

Method 1: Using Zafar Projected Differential Transform Method

Applying the Zafar transformation to equation (95) with respect to τ , we obtain:

$$\frac{s^2}{v^2}Z\{v(\varsigma,\tau)\}-\frac{s^2}{v^2}v(\varsigma,0)-\frac{s}{v}v_\tau(\varsigma,0)=6\varsigma^3\frac{v}{s}-6\varsigma.3!\frac{v^3}{s^3}+\varsigma^6 6!\frac{v^6}{s^6}+Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right] \tag{97}$$

Simplifying the resulting expression gives:

$$Z\{v(\varsigma,\tau)\}=v(\varsigma,0)+\frac{v}{s}v_\tau(\varsigma,0)+6\varsigma^3\frac{v^3}{s^3}-6\varsigma.3!\frac{v^5}{s^5}+\varsigma^4 6!\frac{v^8}{s^8}+\frac{v^2}{s^2}Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right] \tag{98}$$

Incorporating the initial conditions yields:

$$Z\{u(x,t)\}=6x^3.\frac{1}{3!}3!.\frac{u^3}{s^3}-6x.3!..\frac{1}{5!}5!\frac{u^5}{s^5}+x^4.6!.\frac{1}{8!}8!\frac{u^8}{s^8}+\frac{u^2}{s^2}Z\left[u_{xx}(x,t)-(u(x,t))^2\right] \tag{99}$$

Taking the inverse Zafar transform of equation (99), we arrive at:

$$v(\varsigma,\tau)=\varsigma^3\tau^3-3\varsigma.\frac{1}{10}\tau^5+\varsigma^4\frac{1}{56}\tau^8+\frac{v^2}{s^2}Z\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right] \tag{100}$$

To construct the series solution, we apply the Projected Differential Transform Method (PDTM):

$$\sum_{r=0}^{\infty} v_{r+1}(\varsigma,\tau) = Z^{-1}\left\{\frac{v^2}{s^2} Z\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right]\right\} \tag{101}$$

By applying the PDTM, we get:

$$v_{r+1}(\varsigma,\tau) = Z^{-1}\left\{\frac{v^2}{s^2} Z[A_r - B_r]\right\}; \tag{102}$$

Here, A_k and B_k are the projected differential transform of $v_{\varsigma\varsigma}(\varsigma,\tau)$ and the nonlinear source term $(v(\varsigma,\tau))^2$, respectively.

At $r=1$:

$$v_2(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[A_1 - B_1]\right]$$

$$A_1 = [v_1(\varsigma,\tau)]_{\varsigma\varsigma} = (0)_{\varsigma\tau} = 0;$$

$$B_1 = \sum_{m=0}^{r=1} v_m(\varsigma,\tau) v_{r-m}(\varsigma,\tau) = v_0(\varsigma,\tau)v_1(\varsigma,\tau) + v_1(\varsigma,\tau)v_0(\varsigma,\tau) = 0.\varsigma\tau + \varsigma\tau.0 = 0$$

$$v_2(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[0-0]\right] = Z^{-1}[0] = 0 \tag{103}$$

At $r=2$:

$$v_3(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[A_2 - B_2]\right]$$

$$A_2 = [v_2(\varsigma,\tau)]_{\varsigma\varsigma} = (0)_{\varsigma\varsigma} = 0;$$

$$B_2 = \sum_{m=0}^{r=2} v_m(\varsigma,\tau) v_{r-m}(\varsigma,\tau) = 0$$

$$v_3(\varsigma,\tau) = Z^{-1}\left[\frac{v^2}{s^2} Z[0-0]\right] = Z^{-1}[0] = 0 \tag{104}$$

Proceeding recursively, the approximate ZPDTM solution is given by:

$$v(\varsigma,\tau) = \varsigma^3\tau^3 \tag{105}$$

Method 2: Using Laplace Projected Differential Transform Method

Applying the Laplace transformation to equation (95) with respect to τ , we get:

$$s^2 L\{v(\varsigma,\tau)\} - s^2 v(\varsigma,0) - s v_\tau(\varsigma,0) = 6\varsigma^3 \frac{1}{s^2} - 6\varsigma.3!\frac{1}{s^4} + \varsigma^6 6!\frac{1}{s^7} + L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right] \tag{106}$$

Simplifying this equation gives:

$$L\{v(\varsigma,\tau)\} = v(\varsigma,0) + \frac{1}{s} v_\tau(\varsigma,0) + 6\varsigma^3 \frac{1}{s^4} - 6\varsigma.3!\frac{1}{s^6} + \varsigma^4 6!\frac{1}{s^9} + \frac{1}{s^2} L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right] \tag{107}$$

Substituting the initial conditions into the expression results in:

$$L\{v(\varsigma,\tau)\} = 6\varsigma^3.\frac{1}{3!}3!.\frac{1}{s^4} - 6\varsigma.3!..\frac{1}{5!}5!\frac{1}{s^6} + \varsigma^4.6!.\frac{1}{8!}8!\frac{1}{s^9} + \frac{1}{s^2} L\left[v_{\varsigma\varsigma}(\varsigma,\tau) - (v(\varsigma,\tau))^2\right] \tag{108}$$

Taking the inverse Laplace transform leads to:

$$v(\varsigma,\tau)=\varsigma^3\tau^3-3\varsigma.\frac{1}{10}\tau^5+\varsigma^4\frac{1}{56}\tau^8+\frac{1}{s^2}L\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right] \tag{109}$$

Applying the PDTM to this equation yields:

$$\sum_{r=0}^{\infty}v_{r+1}(\varsigma,\tau)=L^{-1}\left\{\frac{1}{s^2}L\left[v_{\varsigma\varsigma}(\varsigma,\tau)-(v(\varsigma,\tau))^2\right]\right\} \tag{110}$$

By applying the PDTM:

$$v_{r+1}(\varsigma,\tau)=L^{-1}\left\{\frac{1}{s^2}L\left[A_r-B_r\right]\right\}; \tag{111}$$

As before, A_k and B_k are the projected differential transform of $v_{\varsigma\varsigma}(\varsigma,\tau)$ and $(v(\varsigma,\tau))^2$ respectively.

At $r=1$:

$$v_2(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}Z\left[A_1-B_1\right]\right]$$

$$A_1=\left[v_1(\varsigma,\tau)\right]_{\varsigma\varsigma}=(0)_{\varsigma\varsigma}=0;$$

$$B_1=\sum_{m=0}^{r=1}v_m(\varsigma,\tau)v_{r-m}(\varsigma,\tau)=v_0(\varsigma,\tau)v_1(\varsigma,\tau)+v_1(\varsigma,\tau)v_0(\varsigma,\tau)=0.\varsigma\tau+\varsigma\tau.0=0$$

$$v_2(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L\left[0-0\right]\right]=L^{-1}\left[0\right]=0 \tag{112}$$

At $r=2$:

$$v_3(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L\left[A_2-B_2\right]\right]$$

$$A_2=\left[v_2(\varsigma,\tau)\right]_{\varsigma\varsigma}=(0)_{\varsigma\varsigma}=0;$$

$$B_2=\sum_{m=0}^{r=2}v_m(\varsigma,\tau)v_{r-m}(\varsigma,\tau)=0$$

$$v_3(\varsigma,\tau)=L^{-1}\left[\frac{1}{s^2}L\left[0-0\right]\right]=L^{-1}\left[0\right]=0 \tag{113}$$

Continuing this process, the LPDTM solution is expressed as:

$$v(\varsigma,\tau)=\varsigma^3\tau^3 \tag{114}$$

Figure 5 displays the exact solutions for the non-homogeneous nonlinear Klein-Gordon model (case 5). Both ZPDTM and LPDTM reproduce the correct analytical behavior across the entire domain.

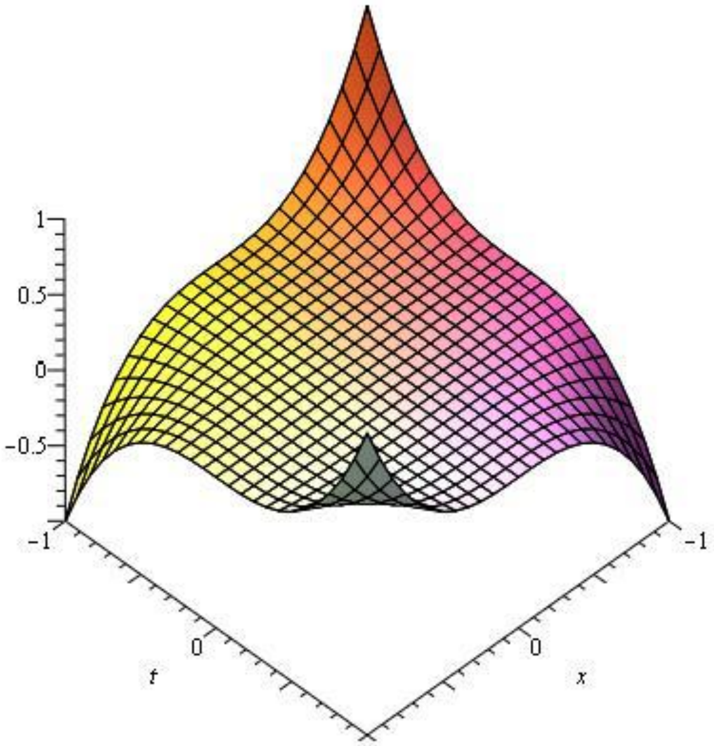

Figure 5: Numerical solution plots for the highly nonlinear non-homogeneous Klein-Gordon equation (case 5) using ZPDTM and LPDTM. The exact match in results highlights the reliability of both approaches in solving stiff nonlinear PDEs without iterative convergence schemes.

CONCLUSION

In this study, we have developed and analyzed two efficient numerical approaches—Zafar Projected Differential Transform Method (ZPDTM) and Laplace Projected Differential Transform Method (LPDTM)—for solving both linear and nonlinear forms of the Klein-Gordon equation. These methods combine the strengths of their respective integral transforms (Zafar and Laplace) with the Projected Differential Transform Method to form direct, non-iterative solution frameworks.

Both ZPDTM and LPDTM demonstrate rapid convergence and high accuracy, which are crucial for minimizing computational effort while preserving numerical precision. Their ability to yield exact or near-exact solutions with fewer series terms highlights a distinct advantage over traditional iterative techniques such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), and Variational Iteration Method (VIM). These conventional methods often require auxiliary constructs or repeated correction steps, which increase complexity and reduce efficiency—especially in nonlinear or stiff problem settings.

The recursive structure of ZPDTM and LPDTM enables systematic computation of solution terms without linearization, making them particularly suitable for physically relevant models in fields such as relativistic mechanics, quantum field theory, and wave propagation. Their reduced computational complexity and strong numerical stability further position them as competitive tools for real-world scientific and engineering applications.

In summary, the proposed methods not only expand the toolkit for addressing Klein-Gordon-type equations but also offer practical advantages in terms of simplicity, scalability, and performance. Their robustness and adaptability suggest significant potential for broader use in computational modeling across applied mathematics and physics.

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CONFLICT OF INTEREST

We declare no conflict of interest in this article

AUTHOR CONTRIBUTION

Sunday O.G: Conceptualization, Methodology, Software, Data collection, Writing original draft.
Adedapo C.L: Visualization, Investigation, Supervision, Software, Validation, Writing.

DATA AVAILABILITY

All data generated or analyzed during this study are included in this published article.

DECLARATION OF GENERATIVE AI

Not applicable.

ETHICS

Not applicable.

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