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The Structural Properties of The Power Graph of Symmetric Groups

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Abstract

This study investigates the power graphs of the symmetric groups S_n for $3 \leq n \leq 6$. In a power graph, vertices represent group elements, and edges connect vertices if one element is a power of the other. The study identifies key structural patterns, including over 250 distinct subgraphs for S_5. Specifically, complete subgraphs K_n, correspond to cyclic subgroups of prime power orders. In contrast, incomplete subgraphs highlight the non-cyclic nature of symmetric groups for $n > 4$, as they cannot be generated by a single element. This research provides a systematic characterization of power graphs for small symmetric groups, offering new insights into their combinatorial structure. Additionally, it contributes to the existing literature by demonstrating how the complexity of power graphs increases with n, revealing intricate power relations and the presence of both cyclic and non-cyclic structural components.

Keywords Power graph of group, symmetric group, cyclic group, non-cyclic group

INTRODUCTION

Various research studies have focused on the power graph of the finite group. Cameron and Ghosh [1] demonstrated that the isomorphism of power graphs implies the isomorphism of the underlying groups, especially for finite abelian group. Chelvam and Sattanathan [2] further explored fundamental characterizations of the power graph of finite abelian group.

The structure of the power graph of certain finite groups has been investigated. For example, Chattopadhayay and Panigrahi [3] studied the power graph of dihedral group, and Asmarani et. all [4] provided more detailed discussion of this graph. Chattopadhayay and Panigrahi [3] also examined the structure of the power graph of dicyclic group, while Mehranian et. all [5] analyzed the power graph of semi dihedral group.

Another important area of research on the power graph of finite groups is its connectivity. The connectivity of the power graph refers to the minimum number of vertices that need to be removed to disconnect the graph. Various approaches to studying the connectivity of the power graph of a group have been presented by Cameron [6], Chattopadhyay and Panigrahi [3], Doostabadi and Ghouchan [7], and Panda and Krishna [8].

Despite the extensive work on power graphs of various finite abelian groups, research on non-abelian groups, particularly symmetric groups, remains sparse. A symmetric group, denoted by Sn , consists of all bijective functions from a finite set to itself, with the composition of functions as the group operation. For $n \geq 3$, symmetric groups are non-abelian and exhibit rich structural complexity. While power graphs of abelian groups have been well-characterized, little attention has been given to non-abelian groups, especially symmetric groups. This presents a significant gap in the existing literature.

In this paper, the power graph of symmetric groups, S_n for $3 \leq n \leq 6$ are explored, with a focus on identifying structural formations. subgraph formations and its properties**.** the study aims to understand how the unique characteristics of symmetric groups influence the structure of their power graphs. This analysis is carried out using a combination of theoretical approaches and computational tools. By addressing this relatively unexplored area, the paper seeks to provide deeper insights into the connectivity and overall structure of the power graphs of symmetric groups.

MATERIALS AND METHODS

This section discusses the introduction concepts and foundational theorems related to the power graph of finite groups. Additionally, notation and definition that will be frequently used throughout this article is also introduced. Let K_n as notation of complete graph with n vertices, which is a simple graph where every pair of distinct vertices are adjacent [9]. Complete graph K_5 is shows by Figure 1. The following definitions from group theory are essential for this study.

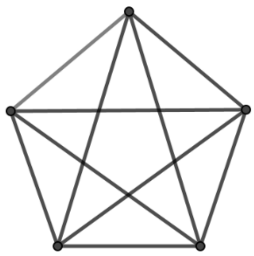

Figure 1. Complete graph K_5

Definition 1 [10] A finite group G is a cyclic group if it can be generated by a single element meaning there is an element $g \in G$ such that $G = \{g^n | \, n \in \mathbb{N}\}$. The element g is referred to as the generator of G. If group not generated by single element, then its noncyclic group.

Definition 2 [11] The symmetric group S_n is the group of all bijections on a finite set of n distinct element. Mathematically, S_n is defined as $S_n = \{\alpha : \{1,2,3,\ldots,n\} \longrightarrow \{1,2,3,\ldots,n\} | \alpha \text{ is bijection}\}$. Here α, β represents a permutation, and the group operation is composition if $\alpha, \beta \in S_n$, then $\alpha \circ \beta \in S_n$.

Let $\alpha \in S_n$, then

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \alpha(1) & \alpha(2) & \alpha(3) & \cdots & \alpha(n) \end{pmatrix}.$$

For example, let $\beta \in S_n$ be a permutation defined by

$$\beta = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 1 & \cdots & n \end{pmatrix},$$

which can be written as $(1, 2, 3)$, with all other elements fixed. This permutation called cycle. More generally, $(a_1, a_2, a_3, \dots, a_k)$ is referred to as a k-cycle or a cycle length k. A cycle (a_1, a_2) is also called transposition. A transposition swaps the elements a_1 and a_2, leaving all other element unchanged.

In the 19th century, Arthur Cayley introduced the concept of visualizing finite groups through graphical representations, now known as Cayley graphs. In these graphs, the vertices represent the elements of the group, while the edges correspond to the actions of the group generators [12]. Building in this concepts, Kelarev and Quinn [13] proposed the concept of power graphs for finite groups and provided a detailed characterization of the structures of power graphs in the context of finite abelian groups.

Definition 3 [13] The power graph of a group $\mathcal{P}(G)$ is a simple graph with vertices representing the set of groups G. It contains edges (u, v) for all u, v in G such that $u \neq v$, and $u^k = v$ or $v^l = u$ for some k and l.

An example of a power graph of groups can be found in the study by Asmarani et al.[4]. They constructed the power graph of dihedral group, D_6. Let $D_6 = \{e, r, r^2, s, rs, r^2 s\}$, then the structure of power graph of D_6 is depicted in Figure 2.

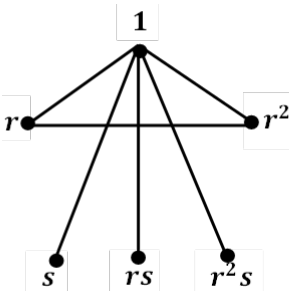

Figure 2. Power graph of D_6

The relationship between cycle groups and the power graph properties of groups is discussed in the following theorem.

Theorem 4 [14], [15] Let G be a finite group. Then $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m with p prime, $m \in \mathbb{N}$.

One of significant properties of power graph of group is its connectivity. The following lemma highlights an importance characteristic of the connectivity of power graph of finite group.

Lemma 5 [15] If G be a finite group then $\mathcal{P}(G)$ is always connected.

Since the identity element in the power graph of a finite group $\mathcal{P}(G)$ is adjacent to all other vertices in $\mathcal{P}(G)$, then $\mathcal{P}(G)$ is always connected.

The following section presents the algorithm for identifying and classifying the subgraph types within the power graph of a symmetric group. Specifically, the algorithm distinguishes between complete subgraphs and incomplete subgraphs, which correspond to different structural relationships within the power graph of a symmetric group. The detailed steps of the algorithm are outlined as follows:

1. Determine the elements of the symmetric group. Enumerate all elements of the symmetric group S_nThe total number of elements is $n!$, representing all possible permutations in S_n.
2. Map the elements to vertices. Assign each element of the symmetric group to a unique vertex in the power graph of S_n. Thus, the vertices of the power graph correspond to the permutations within S_n.
3. Establish adjacency relationships. For each pair of vertices a, b in S_n, define an edge between them if and only if $a = b^k$ or $b = a^k$ for some integer k. This adjacency criterion adheres to the definition of the power graph, where one element is a power of the other.
4. Identify subgraph structures. Detect complete subgraphs that correspond to k-cycle permutations within the symmetric group. These subgraphs represent subsets of elements that form cyclic subgroups of S_n.
5. Identify incomplete subgraphs that arise from permutations that not correspond to k-cycle, reflecting the more complex relationships between non-cyclic elements of the group.

Once all subgraphs within the power graph of symmetric groups are identified, their respective properties are systematically classified. This classification aims to provide a deeper understanding of the overall structure and complexity of the power graph of symmetric groups.

RESULTS AND DISCUSSION

The structure of power graphs of symmetric groups is introduced with the following lemma.

Lemma 6 (Vertex and edge properties of power graph of symmetric group)

Let S_n be the symmetric group of degree n, with $3 \leq n \leq 6$. The power graph of symmetric group $\mathcal{P}(S_n)$ has the following properties:

i. The vertices of $\mathcal{P}(S_n)$ correspond to the element of symmetric group S_n, with a total of $n!$ vertices.
ii. An edge exists between two vertices σ and τ in $\mathcal{P}(S_n)$ if and only if $\sigma = \tau^m$ or $\tau = \sigma^m$ for some integer m.

This lemma can be directly proven using the definition of power graph of finite groups. Next, Theorem 6 discusses the structural properties of the power graph for the symmetric group.

Theorem 7 (The structural properties of power graph of symmetric group)

Let S_n be the symmetric group of degree n, with $3 \leq n \leq 6$. The power graph of symmetric group, $\mathcal{P}(S_n)$, has the following properties:

i. For each $k \leq n$, there is exist a complete subgraph K_k in $\mathcal{P}(S_n)$ with the identity element corresponding to k-cycles.
ii. Complete subgraphs are formed by cyclic subgroups of prime power order.
iii. The identity element is connected to every other element in $\mathcal{P}(S_n)$
iv. The power graph $\mathcal{P}(S_n)$ is connected.

Proof:

i. The set of k-cycles and identity element form a subgraph of $\mathcal{P}(S_n)$. Since each element in set of k-cycles and the identity element can be expressed as power of each other, then every pair of distinct vertices is connected. Consequently, k-cycles and the identity element together form a compete graph K_k in $\mathcal{P}(S_n)$.

ii. Subgraph in $\mathcal{P}(S_n)$ that formed by k-cycles together with identity element are complete graph. Based on Theorem 4, is a cyclic subgroups of prime power order.
iii. Let e be the identity element in $\mathcal{P}(S_n)$. Since $\mathcal{P}(S_n)$ is a power graph then $k^m = e$ for each $k \in S_n$ and some $m \in \mathbb{N}$. Therefore, e is connected to each $k \in S_n$ with edges in $\mathcal{P}(S_n)$.
iv. This part is directly proven by property in Theorem 6 iii.

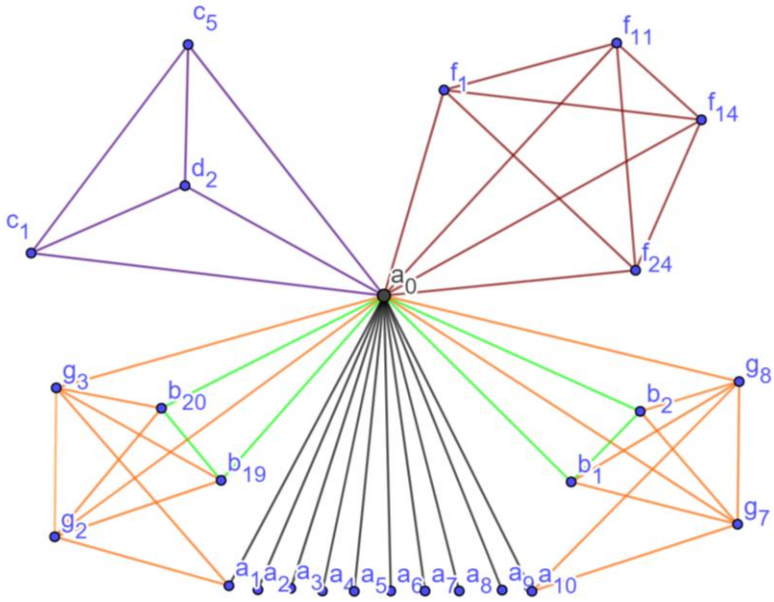

Figure 3. The subgraph of The Power Graph of S_5

Let $S_5 = \{a_0, a_1, a_2, \dots, a_{10,} b_1, \dots, b_{20}, c_1, \dots, c_{30}, d_1, \dots, d_{15,} f_1, \dots, f_{24}, g_1, \dots, g_{20}\}$. There are 120 vertices on the power graph of S_5. Figure 3 shows a subgraph of S_5 which illustrates various of complete and incomplete subgraphs in the power graph of S_5. Each 2-cycle vertex, along with the identity vertex, forms a complete graph K_2 (depicted in black). Pairs of 3-cycle vertices, together with the identity vertex, form a complete graph K_3 (depicted in green), and there are ten such K_3 subgraphs. Additionally, pairs of 4-cycle vertices and pairs of transposition vertices, along with the identity vertex, form a complete graph K_4 (depicted in purple), with ten such K_4 subgraphs present. Four 5-cycle vertices, along with the identity element, form a complete graph K_5 (depicted in maroon), and there are six K_5 subgraphs. The final subgraph type, shown in orange, is not a complete graph. It is formed by a combination of an orbit of length 3 and a transposition, which generates 3-cycle vertices and transpositions. There are 10 such incomplete graphs.

Now, let $S_6 = \{a_0, a_1, a_2, \dots, a_{15,} b_1, \dots, b_{40}, c_1, \dots, c_{90}, d_1, \dots, d_{144,} f_1, \dots, f_{135}, k_1, \dots, k_{60}, k_1, \dots, k_{60}, h_1, \dots, h_{10}, j_1, \dots, j_{120}, m_1, \dots, m_{90}, n_1, \dots, n_{10}\}$. The structure of the power graph of the symmetric group S_6 is significantly complex, with numerous vertices and subgraph. Both complete and incomplete graphs are present, highlighting the intricate relationships between elements in S_6. Over 250 distinct subgraphs are contained within the power graph of S_6, contributing to the complexity of its structure. Some subgraphs of the power graph of S_6 are presented in Figure 4, 5 and 6.

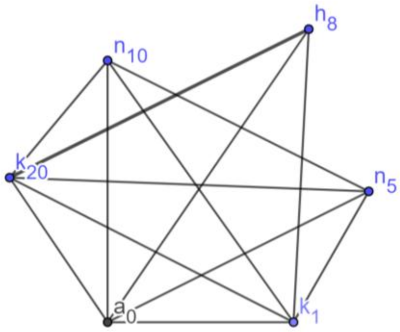

Figure 4. Subgraph of The Power Graph of S_6

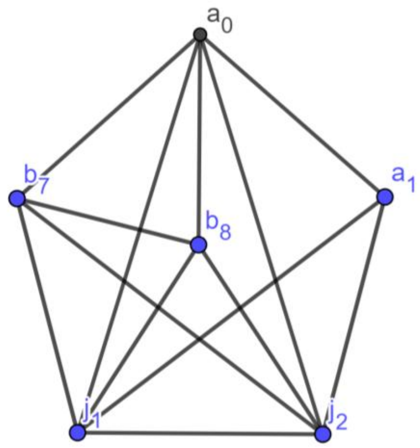

Figure 5. Subgraph of The Power Graph of S_6

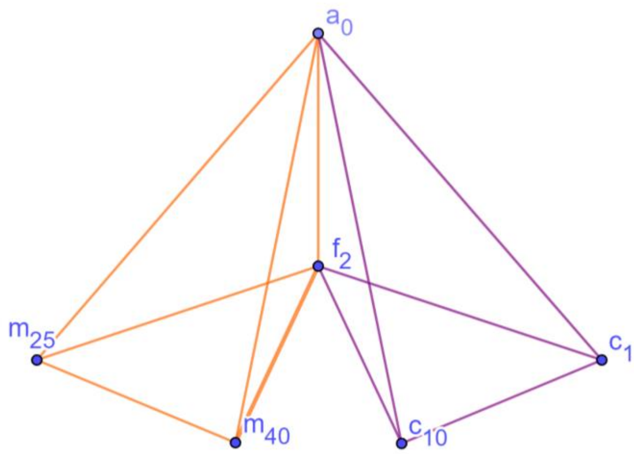

Figure 6. Complete Subgraph of The Power Graph of S_6

Figure 4 present a subgraph for the power relation of 6-cycle and product of orbit. The subgraph is not a complete subgraph. There are 30 such subgraphs within the power graph of S_6. Figure 5 present a subgraph for the power relation of orbits. The subgraph is not a complete subgraph. There are 60 such subgraphs within the power graph of S_6. Figure 6 presents a subgraph illustrating the power relationship between orbits and transpositions (depicted in orange). Additionally, it shows the power relationship of 4-cycles and the composition of transpositions (depicted in purple). The subgraph is complete graph K_4.

Based on the previously discussed structure of the power graph of symmetric groups, it is evident that these graphs exhibit significantly greater complexity when compared to the power graphs of other groups. For instance, in contrast to the power graph of dihedral groups and dicyclic groups, both of which contain only one complete subgraph while the remaining subgraphs are incomplete [3], the power graph of symmetric groups demonstrates a more intricate and diverse structural composition. This emphasizes the unique characteristics and higher structural complexity of the power graphs of symmetric group.

CONCLUSION

The study examined the power graph of symmetric groups S_n for $3 \leq n \leq 6$, focusing on their structure, subgraph formations, and connectivity. The finding reveal that the power graph of symmetric groups consists of complete and incomplete subgraphs, reflecting the complex and non-cyclic nature. Various subgraphs, such as complete graphs K_k corresponding to k-cycles, demonstrate the intricate relationships between elements within the graph.

As n increases, the complexity of the power graph grows, highlighted by identification of over 250 distinct subgraphs in the power graph of S_6. These findings contribute to a deeper understanding of the algebraic characteristics of symmetric groups and their graphical representations, showing the rich structure of the power graph.

In addition, future research could focus on generalizing the structure of power graphs for symmetric groups to all n. This would provide a more comprehensive understanding of the structural patterns and its properties across symmetric groups of arbitrary degrees. Furthermore, exploring power graphs for other non-abelian groups would expand the applicability of these findings and contribute to a deeper understanding of their algebraic and graphical properties.

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