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Exploring the Efficacy of the Weighted Average Method for Solving Nonlinear Partial Differential Equations: A Study on the Burger-Fisher Equation

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Abstract

This study explores the application of the weighted average method for solving the Burger-Fisher equation, a nonlinear partial differential equation (PDE) of significant interest in various scientific disciplines. Nonlinear PDEs, such as the Burger-Fisher equation, are fundamental in describing complex physical, biological, and engineering phenomena but pose challenges for both analytical and numerical solutions. The weighted average method, known for its ability to converge rapidly to exact solutions, offers a promising approach for tackling such equations. By discretizing both spatial and temporal derivatives using a combination of forward, backward, and central differences, the method approximates solutions with high accuracy and stability. Conducting convergence and stability analyses, this study elucidates the computational requirements and performance characteristics of the weighted average method. Utilizing mathematical software like MATLAB and MAPLE, the method's implementation involves solving a tridiagonal matrix system at each time step. Comparison between numerical solutions obtained using the method and exact solutions demonstrates the method's accuracy, with negligible errors observed. Visual representations further illustrate the close agreement between the numerical and exact solutions, validating the method's reliability for practical applications. The study's findings underscore the practical utility of the weighted average method in solving the Burger-Fisher equation and similar nonlinear PDEs. Its ability to accurately approximate solutions while maintaining stability highlights its efficacy as a computational tool for addressing complex mathematical problems across diverse scientific and engineering fields. The study contributes to advancing the understanding and application of numerical methods for nonlinear PDEs, offering valuable insights for researchers and practitioners seeking precise and reliable solutions to complex mathematical models. Overall, the study emphasizes the importance of fine-tuning numerical parameters and leveraging computational resources to achieve optimal accuracy when utilizing the weighted average method for solving nonlinear PDEs.

Keywords: Burger-Fisher equation, Weighted Average Method, Nonlinear Partial Differential equation

INTRODUCTION

Nonlinear partial differential equations (PDEs) are a class of mathematical equations that involve unknown functions of multiple variables and their partial derivatives, where the relationship between these functions and their derivatives is nonlinear. These equations are fundamental in describing a wide array of physical, biological, and engineering phenomena, characterized by their complexity and the challenge they pose for both analytical and numerical solutions. Nonlinear PDEs are crucial for modeling various processes in science and technology [1-5]. They appear in diverse fields such as fluid dynamics, plasma physics, solid

mechanics, and quantum field theory. In fluid dynamics, for example, they describe the behavior of fluids under different conditions, capturing phenomena like turbulence and shock waves. In biological systems, nonlinear PDEs can model population dynamics, the spread of diseases, and pattern formation in developmental biology. In chemistry, they are used to describe reaction-diffusion systems, which are essential for understanding processes like chemical reactions and the spread of substances [6-10].
The general form of a nonlinear PDE can be written as $(u,\ u_x,\ u_{t,,,t}, \dots\ ) = 0$, Where P is a polynomial involving the function (u) and its partial derivatives. Solving these equations analytically or numerically is essential for creating accurate models of physical systems.

Finding exact solutions to nonlinear PDEs is often challenging due to their complexity. Analytical methods are limited to special cases and often require sophisticated mathematical techniques. As a result, numerical methods play a vital role in solving these equations. Common numerical methods include finite difference, finite element, and finite volume methods, which discretize the equations and solve them approximately on a computational grid. One effective numerical approach is the weighted average method, which discretizes both spatial and time derivatives. This method is noted for its ability to converge rapidly to the exact solution, making it a valuable tool in computational simulations [10-15]. The Burgers-Fisher equation is a prominent example of a nonlinear PDE, combining elements from both the Burgers and Fisher equations. It models phenomena involving convection, diffusion, and reaction processes, such as fluid dynamics, heat conduction, and population genetics. This equation is particularly useful for studying the interaction between reaction mechanisms, convection effects, and diffusion transport. Another significant application of nonlinear PDEs is in reaction-diffusion systems, which describe the spatial and temporal evolution of chemical concentrations. These systems are used to model processes such as pattern formation in biological organisms and the spread of chemical reactions [15-20]. The study of nonlinear PDEs is a dynamic and ongoing field of research. One of the main challenges is developing efficient and accurate numerical methods to solve these equations for complex systems. Researchers are also focused on finding more general analytical solutions and understanding the underlying structures and behaviors of these equations. In recent years, advanced computational techniques, including machine learning and high-performance computing, have been applied to the study of nonlinear PDEs, offering new avenues for exploration and solution strategies. These methods have the potential to uncover new insights and provide more accurate models for the complex phenomena described by nonlinear PDEs [25-30].

The Burgers equation is a fundamental partial differential equation that plays a crucial role in various fields of applied mathematics and physics. It is named after the Dutch physicist Johannes Martinus Burgers, who introduced it as a simplified model for turbulent fluid flow. The Burgers equation is widely used to model various physical phenomena. In fluid dynamics, it describes the propagation of shock waves and turbulence in a viscous medium. It is also applied in traffic flow modeling, the density of cars, and in gas dynamics, where it helps understand the behavior of gas particles under certain conditions. Additionally, the equation finds use in acoustics, where it models nonlinear wave propagation in dissipative media. The Burgers equation combines nonlinear convection and linear diffusion terms. This combination captures the essential features of more complex fluid dynamic equations like the Navier-Stokes equations [31-36]. The nonlinear term represents the inertial forces, while the diffusion term accounts for the viscous effects. This dual nature makes the Burgers equation a valuable tool for studying the interplay between nonlinearity and diffusion in physical systems. For specific initial and boundary conditions, the Burgers equation can be solved exactly using methods such as the Cole-Hopf transformation, which transforms it into a linear heat equation. This solution provides insights into the formation and evolution of shock waves and other complex structures. Due to its simplicity and rich structure, the Burgers equation is often used as a benchmark for testing numerical methods. Techniques like finite difference, finite element, and spectral methods are commonly employed to solve it numerically [37-42]. These methods discretize the equation in time and space, allowing for the simulation of its dynamic behavior under various conditions. The Burgers equation serves as a prototypical example for understanding more complex nonlinear PDEs. Its study has led to the development of various analytical and numerical techniques, which are applicable to a wide range of problems in applied mathematics and physics. Furthermore, it helps researchers explore fundamental concepts such as shock wave formation, turbulence, and the effects of nonlinearity and diffusion [43-49].

The Fisher equation, also known as the Fisher-Kolmogorov equation or the Fisher-KPP equation, is a key nonlinear partial differential equation that models the spread of advantageous genes in a population. Named after the British statistician and geneticist Ronald Fisher, who introduced it in 1937. The Fisher equation is fundamental in various fields, particularly in biology and ecology. It describes the spatial spread of advantageous genetic traits, invasive species, and diseases. The equation is also relevant in chemical kinetics, where it models the reaction-diffusion processes, and in the spread of cultural or technological innovations. The Fisher equation combines linear diffusion and logistic growth. The diffusion term models the random movement or dispersal of individuals in space, while the logistic growth term captures the reproduction rate and carrying capacity constraints [49-54]. This combination leads to traveling wave solutions that describe how an advantageous gene or species spreads spatially over time. The Fisher equation has traveling wave solutions. These solutions are typically obtained using techniques like the method of characteristics or perturbation methods. Numerical methods are extensively used to solve the Fisher equation, especially for complex initial and boundary conditions where analytical solutions are not feasible. Common numerical techniques include finite difference, finite element, and spectral methods. These approaches discretize the equation in space and time, allowing for the simulation of wave propagation and the dynamics of population spread. The Fisher equation has profound implications in theoretical and applied research. In population genetics, it provides insights into how advantageous traits propagate through a population, influencing evolutionary dynamics. In ecology, it helps understand the spread of invasive species and the impact of habitat fragmentation [55-60]. The equation also finds applications in epidemiology, where it models the spatial spread of infectious diseases. Beyond its biological origins, the Fisher equation is applicable in various other domains. In neuroscience, it models the spread of electrical activity in neural tissues. In environmental science, it describes pollutant dispersion in ecosystems. Its versatility and foundational nature make it a critical tool in modeling and understanding diffusion-reaction processes across different scientific fields [61-63].

The Burgers-Fisher equation is a notable nonlinear partial differential equation that integrates elements from both the Burgers equation and the Fisher equation. The Burgers-Fisher equation is utilized in a variety of fields, including fluid dynamics, biological population dynamics, and chemical kinetics. In fluid dynamics, it models the combined effects of convection, diffusion, and reaction, capturing complex behaviors such as shock wave formation and turbulence. In ecology, it describes the spread of a species with both dispersal and logistic growth. This equation combines the nonlinear convective term from the Burgers equation with the diffusive term and the logistic growth term from the Fisher equation [62-66]. The convective term represents the transport due to velocity; the diffusive term models the spreading due to viscosity or random movement, and the logistic growth term accounts for population increase and environmental carrying capacity. Solving the Burgers-Fisher equation analytically is challenging due to its nonlinear nature. However, specific cases can be approached using transformation methods or perturbation techniques. Numerically, methods such as finite difference, finite element, and spectral methods are employed. These techniques discretize the equation, facilitating the study of its dynamic behavior under various initial and boundary conditions. The Burgers-Fisher equation serves as a prototypical model for understanding the interaction between convection, diffusion, and reaction mechanisms. It helps researchers investigate complex phenomena like shock wave formation, turbulence, and population dynamics. The equation is a benchmark for testing numerical methods and exploring the interplay of nonlinear and diffusive effects in various scientific and engineering applications [4, 7-10, 60-66].

The Crank-Nicolson method is a widely used numerical scheme for solving time-dependent partial differential equations, particularly those involving parabolic PDEs like the heat equation. Named after John Crank and Phyllis Nicolson, who introduced it in 1947, this method is a finite difference approach that is implicit in time and second-order accurate in both time and space. The Crank-Nicolson method averages the implicit backward Euler method and the explicit forward Euler method. It achieves this by taking the average of the PDE at the current time step and the next time step. The Crank-Nicolson method is unconditionally stable for linear problems, meaning it does not require a restrictive relationship between the time step size and the spatial grid size. Its second-order accuracy in both time and space makes it more accurate than purely explicit or implicit methods. However, for nonlinear problems, the method may require iterative techniques, such as Newton's method, to solve the resulting system of equations at each time step. Due to its stability and accuracy, the Crank-Nicolson method is extensively used in various fields. In finance, it is applied to option pricing models, such as the Black-Scholes equation. In engineering, it is used for simulating heat conduction and diffusion processes. Its ability to handle stiff equations efficiently makes it suitable for numerous scientific and engineering applications. To implement the Crank-Nicolson method, one typically discretizes the spatial domain into a grid and applies the method to each grid point, resulting in a system of linear equations that must be solved at each time step. This often involves using matrix solvers or iterative methods to handle the implicit nature of the scheme [30-34, 50-55].

The weighted average method is a numerical technique used to solve partial differential equations (PDEs) by discretizing the spatial and temporal derivatives. This method is particularly effective for equations that model physical phenomena involving diffusion, convection, and reaction processes. In the weighted average method, the PDE is discretized using finite differences, where the derivatives are approximated by differences between function values at discrete grid points. The method employs a combination of forward, backward, and central differences to approximate the derivatives, assigning weights to these differences to balance accuracy and stability. The key steps in implementing the weighted average method are discretization where the domain of the PDE is divided into a grid of points in both space and time. The spatial derivative is approximated using a weighted average of forward and backward differences, while the temporal derivative can be handled using an implicit or explicit scheme. The weights are chosen to optimize the stability and accuracy of the method. The discretized PDE is transformed into a system of algebraic equations that can be solved using numerical linear algebra techniques. The weighted average method is widely used for solving nonlinear PDEs such as the Burgers-Fisher equation, where it efficiently captures the interaction between convection, diffusion, and reaction terms. It is particularly useful in fluid dynamics, heat transfer, and other areas where high accuracy and stability are required. By appropriately choosing weights, the method can achieve high accuracy in approximating derivatives also the method is robust and can handle stiff problems, where traditional methods might fail. It can be adapted to various types of PDEs and boundary conditions [19-25, 33-37].

Researchers have employed various methods to tackle the Burgers-Fisher equation, including integral transforms, wavelet Galerkin methods, finite difference schemes, and pseudospectral methods. These approaches aim to obtain accurate solutions and explore the equation's behavior under different conditions. Additionally, new algorithms and heuristic schemes have been proposed to address the challenges posed by the generalized Burger-Fisher equation, further advancing the understanding and application of this fundamental model [20-40].

However, this study introduces and explores the application of the weighted average method for solving the Burger-Fisher equation, a nonlinear partial differential equation (PDE) of considerable interest in various scientific disciplines. By employing this numerical technique, the research contributes to expanding the toolbox of methods available for tackling nonlinear PDEs. The study advances the understanding and application of the weighted average method, a numerical approach that involves discretization of PDEs using forward, backward, and central differences. Through convergence and stability analyses, the research elucidates the efficacy and computational requirements of this method, providing valuable insights for future studies in numerical analysis and computational mathematics. The comparison of results between the weighted average method and exact solutions demonstrates the accuracy and reliability of the proposed approach in approximating solutions to the Burger-Fisher equation. The small errors observed indicate that the method yields highly accurate results, making it suitable for practical applications where precision is essential. The study's findings have practical implications for researchers and practitioners working in fields where the Burger-Fisher equation arises, such as fluid dynamics, population dynamics, and reaction-diffusion systems. The ability to accurately solve this equation using the weighted average method enhances the understanding of complex phenomena and facilitates the development of predictive models for real-world scenarios. Despite its high computational power requirements, the weighted average method offers a computationally efficient approach for solving nonlinear PDEs like the Burger-Fisher equation. By leveraging mathematical packages like MATLAB and MAPLE, researchers can effectively implement this method to obtain numerical solutions with reasonable computational resources. The insights gained from this study extend beyond the specific problem of solving the Burger-Fisher equation and can be applied to a wide range of nonlinear PDEs encountered in diverse scientific and engineering disciplines. The principles and methodologies developed in this research serve as a foundation for addressing similar challenges in other contexts.

MATERIALS AND METHODS

1.1. Basic governing equation

The generalized form of the Burgers-Fisher equation is given as:

$$\frac{\partial u}{\partial t}+\alpha u^{\delta}\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=\beta u(1-u^{\delta}) \tag{1}$$

When $\delta=1$, equation (1) becomes;

$$\frac{\partial u}{\partial t}-\alpha u\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=\beta u(1-u) \tag{2}$$

Which can be written as;

$$U_t+\alpha uU_x-U_{xx}=(1-U) \tag{3}$$

2.0 SOLUTION OF BURGER-FISHER EQUATION

2.1. The Exact Solution of the Burger-Fisher Equation

Consider the Burger-Fisher equation in (1) subject to the initial condition

$$u(x,0)=\left[\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{\alpha\delta}{2(\delta+1)}x\right)\right]^{\frac{1}{\delta}} \tag{4}$$

and the exact solution is given by

$$u(x,t)=\left[\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{\alpha\delta}{2(\delta+1)}\left(x-\left(\frac{\alpha}{\delta+1}+\frac{\beta(\delta+1)}{\alpha}\right)t\right)\right)\right] \tag{5}$$

Note that for $\delta=1$, the exact solution is

$$u(x,t)=\left[\frac{1}{2}+\frac{1}{2}\tanh\left(-\frac{\alpha}{4}\left(x-\left(\frac{\alpha}{2}+\frac{2\beta}{\alpha}\right)t\right)\right)\right] \tag{6}$$

2.2. The $\boldsymbol{\theta}$-scheme:

Suppose we have a Partial Differential Equation

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}\Rightarrow\frac{\partial u}{\partial t}=-\frac{\partial^2 u}{\partial x^2}, \tag{7}$$

The θ-scheme for the above is given as;

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=-\left[\frac{u_{i-1}^n+2u_i^n-u_{i+1}^n}{(\Delta x)^2}\right] \tag{8}$$

So we have

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=-\left[\frac{u_{i-1}^n+2u_i^n-u_{i+1}^n}{(\Delta x)^2}\right]\left[\theta u_i^n+(1-\theta)u_i^{n+1}\right] \tag{9}$$

There are three most used values of θ, i. e, $0, 0.5, 1$

For $\theta = 0$, we have the Forward Euler Method:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{(\Delta x)^2}\left[u_{i-1}^n + 2u_i^n - u_{i+1}^n\right]u_i^{n+1} \tag{10}$$

For $\theta = 1$, we have the Backward Euler Method:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{(\Delta x)^2}\left[u_{i-1}^n + 2u_i^n - u_{i+1}^n\right]u_i^n \tag{11}$$

For $\theta = \frac{1}{2}$, we have the Crank-Nicolson Method:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{(\Delta x)^2}\left[u_{i-1}^n + 2u_i^n - u_{i+1}^n\right]\left[\frac{1}{2}u_i^n + \frac{1}{2}u_i^{n+1}\right] \tag{12}$$

Which is also known as the weighted average Method.

The Crank-Nicolson scheme was introduced by John Crank and Phyllis Nicolson in 1947. It is a finite difference scheme which is used to approximate the solution of nonlinear differential equation. It is sometimes called a semi-implicit method. The Crank-Nicolson scheme uses the weighted average of half of the higher order derivative calculated at n+1time level and the other half at n time level. The solution of the Crank-Nicolson method involves a system of tridiagonal matrix to be solved at each time level. The Crank-Nicolson method is a competitive algorithm for the numerical solution of one-dimensional problems for the heat equation. It requires great computational power and its calculation is carried out using mathematical packages like MAPLE, MATLAB.

2.3. The Weighted Average Method for Burgers-Fisher Equation

The weighted average method is a method that involves discretization of the equation being considered using the forward difference, backward difference and central difference. The weighted average method is a numerical method which has been said to converge faster to the exact solution than the other analytical methods. It is a difference method that involves discretizing the spatial derivatives and the time derivative.

For the numerical value approximation using the weighted average method, the following notations are used: Δx to represent the spatial step length and Δt to represent the time step length. The numerical solution of $u(x,t) = u_i^n$, where $x = x_j = x_0 + j\Delta x$ and $t = t_n + n\Delta t$

Consider the Burger-Fisher equation given in (2), to discretize the equation, we use the following procedure:

1. Central difference

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{13}$$

2. Forward difference

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} \tag{14}$$

Discretizing u_t by forward difference and u_{xx} *and* u_x by central difference

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \alpha u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}\right) = \beta u_i^{n+1}(1 - u_i^n) \tag{15}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}\right] - \alpha u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \beta u_i^{n+1}(1 - u_i^n) \tag{16}$$

$$u_i^{n+1} - u_i^n = \frac{\Delta t}{(\Delta x)^2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right] - \frac{\Delta t \alpha u_i^n}{2\Delta x}\left[u_{i+1}^{n+1} - u_{i-1}^{n+1}\right] + \Delta t \beta u_i^{n+1}(1 - u_i^n) \tag{17}$$

Discretizing u_t by forward difference and u_{xx} *and* u_x by central difference explicitly

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} - \alpha u_i^n \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \beta u_i^{n+1}(1 - u_i^n) \tag{18}$$

$$u_i^{n+1} - u_i^n = \frac{\Delta t}{(\Delta x)^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right] - \frac{\Delta t \alpha u_i^n}{2\Delta x}\left[u_{i+1}^{n+1} - u_{i-1}^{n+1}\right] + \Delta t \beta u_i^{n+1}(1 - u_i^n) \tag{19}$$

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{(\Delta x)^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right] - \frac{\Delta t \alpha u_i^n}{2\Delta x}\left[u_{i+1}^{n+1} - u_{i-1}^{n+1}\right] + \Delta t \beta u_i^{n+1}(1 - u_i^n) \tag{20}$$

The weighted average of (4) and (5) gives the weighted average scheme of the Burger-Fisher equation. That is, (4) + (1 − θ) (5)

Where $\theta \in [0,1]$

Thus,

$$u_i^{n+1} = u_i^n + \Delta t \beta u_i^{n+1}(1 - u_i^n) - \frac{\Delta t \alpha u_i^n}{2\Delta x}(u_{i+1}^{n+1} - u_{i-1}^{n+1}) + \frac{\Delta t}{(\Delta x)^2}\left[\theta(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + (1-\theta)(u_{i+1}^n - 2u_i^n + u_{i-1}^n)\right] \tag{21}$$

If $\theta = \frac{1}{2}$, the scheme becomes;

$$u_i^{n+1} = u_i^n + \Delta t \beta u_i^{n+1}(1-u_i^n) - \frac{\Delta t \alpha u_i^n}{2\Delta x}(u_{i+1}^{n+1} - u_{i-1}^{n+1}) + \frac{\Delta t}{(\Delta x)^2}\left[\frac{1}{2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)\right] \quad (22)$$

We then obtain

$$u_i^{n+1} - \Delta t \beta u_i^{n+1} + \Delta t \beta u_i^{n+1} u_i^n + \frac{\Delta t \alpha u_i^n}{2\Delta x}[u_{i+1}^{n+1} - u_{i-1}^{n+1}] - \frac{\Delta t}{2(\Delta x)^2} u_{i+1}^{n+1} + \frac{\Delta t}{(\Delta x)^2} u_i^{n+1} - \frac{\Delta t}{2(\Delta x)^2} = u_i^n + \frac{\Delta t}{2(\Delta x)^2} u_{i+1}^n - \frac{\Delta t}{(\Delta x)^2} u_i^n + \frac{\Delta t}{2(\Delta x)^2} u_{i-1}^n \quad (23)$$

Multiplying (23) by 2, we have,

$$2u_i^{n+1} - 2\Delta t \beta u_i^{n+1} + \Delta t \beta u_i^n + \frac{\Delta t \alpha u_i^n}{2\Delta x}(u_{i+1}^{n+1} - u_{i-1}^{n+1}) - \frac{\Delta t}{(\Delta x)^2} u_{i+1}^{n+1} + \frac{2\Delta t}{(\Delta x)^2} u_i^{n+1} - \frac{\Delta t}{(\Delta x)^2} = 2u_i^n + \frac{\Delta t}{(\Delta x)^2} u_{i+1}^n - \frac{2\Delta t}{(\Delta x)^2} u_i^n + \frac{\Delta t}{(\Delta x)^2} u_{i-1}^n \quad (24)$$

$$2u_i^{n+1}(1 - \beta\Delta t + \beta u_i^n \Delta t + r) - r(u_{i+1}^{n+1} + u_{i-1}^{n+1}) + \alpha p u_i^n (u_{i+1}^{n+1} - u_{i-1}^{n+1}) = 2u_i^n(1-r) + r(u_{i+1}^n + u_{i-1}^n) \quad (25)$$

Where $r = \frac{\Delta t}{(\Delta x)^2}$, $p = \frac{\Delta t}{\Delta x}$

2.4. Stability Analysis

To perform the stability analysis of this weighted average method for the Burger-Fisher equation, Von Neumann stability analysis will be used. However, the Von Neumann stability analysis is applicable to linear partial differential equation. Since the Burger-Fisher equation is a nonlinear partial differential equation, it has to be linearized.

Recall that the Burger-Fisher equation is given as

$$\frac{\partial u}{\partial t} - \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u)$$

To linearize this, the nonlinear term $(1-u)$ will be expanded by Taylor series expansion around an equilibrium state say U ,i.e. $u = U$, where u is a constant. So we have,

$$\beta u(1-u) = \beta U(1-U) + \beta(U-u) - \frac{\beta}{2}(U-u)^2 \quad (26)$$

Substituting into the Burger-Fisher equation, we have

$$\frac{\partial u}{\partial t} - \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x} = \beta U(1-U) + \beta(U-u) - \frac{\beta}{2}(U-u)^2 \quad (27)$$

Neglecting higher order terms, we have,

$$\frac{\partial u}{\partial t} - \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta U(1-U) + \beta(U-u) \tag{28}$$

Using the difference equation, that is;

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \alpha u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} = \beta U(1-U) + \beta(U - u_i^n) \tag{29}$$

By simple cross multiplication, we have;

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - \alpha u_i^n \frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \beta U(1-U) + \beta(U - u_i^n) \tag{30}$$

If $U = 0$, the have;

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - \alpha u_i^n \frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \beta u_i^n \tag{31}$$

We define the error approximation as;

$$\varepsilon_i^n = u_i^n - u_i^n \,,\ \varepsilon_i^{n+1} = \varepsilon_i^{n+1} + \frac{\Delta t}{(\Delta x)^2}({}_{i+1}^{n} - 2\varepsilon_i^n + \varepsilon_{i-1}^n) - \alpha\varepsilon_i^n \frac{\Delta t}{2\Delta x}(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n) - \beta\varepsilon_i^n \tag{32}$$

And by Fourier transform, we have that;

$$\varepsilon(x,\ t) = \sum_{m=-M}^{m=M} E_m(t) e^{ik_m x} \text{ where } k_m = \frac{m\pi}{L} \tag{33}$$

Hence,

$$\varepsilon_i^n = E_m(t)e^{ik_m x},\ \ \varepsilon_{i+1}^n = E_m(t)e^{ik_m(x+\Delta x)},\ \ \varepsilon_{i-1}^n = E_m(t)e^{ik_m(x-\Delta x)},\ \ \varepsilon_i^{n+1} = E_m(t\Delta t)e^{ik_m x}, \tag{34}$$

$$\begin{aligned} E_m(t+\Delta t)e^{ik_m x} &= E_m(t)e^{ik_m x} + \frac{\Delta t}{(\Delta x)^2}\left[E_m(t)e^{ik_m(x+\Delta x)} - 2E_m(t)e^{ik_m x} + E_m(t)e^{ik_m(x-\Delta x)}\right] \\ &\quad + \alpha u_i^n \frac{\Delta t}{2\Delta x}\left[E_m(t)e^{ik_m(x+\Delta x)} - E_m(t)e^{ik_m(x-\Delta x)}\right] - \beta E_m(t)e^{ik_m x} \end{aligned} \tag{35}$$

Dividing all through by e^{ikmx} gives;

$$\begin{aligned} E_m(t+\Delta t) &= E_m(t) + \frac{\Delta t}{(\Delta x)^2}[E_m(t)e^{ik_m\Delta x} - 2E_m(t) + E_m(t)e^{-ik_m\Delta x}] + \alpha u_i^n \frac{\Delta t}{2\Delta x}[E_m(t)e^{ik_m\Delta x} - E_m(t)e^{-ik_m\Delta x}] - \beta E_m(t) \\ &= E_m(t)\left[1 + \frac{\Delta t}{(\Delta x)^2}e^{ik_m\Delta x} - 2\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{(\Delta x)^2}e^{-ik_m\Delta x}\right] + E_m(t)\left[\frac{\alpha u_i^n \Delta t}{2\Delta x}e^{ik_m\Delta x} - \frac{\alpha u_i^n}{2\Delta x}e^{-ik_m\Delta x} - \beta\right] \\ &= 1 + \Delta t\left[\frac{e^{ik_m\Delta x} + e^{-ik_m\Delta x}}{(\Delta x)^2} + \alpha u_i^n\left(\frac{e^{ik_m\Delta x} - e^{-ik_m\Delta x}}{2\Delta x}\right) - \beta\right] \end{aligned} \tag{36}$$

Using the identity $\sin x = e^x - e^{-x}$, the above becomes;

$$\frac{E\ \ (t+\Delta t)}{E_m(t)} = 1 + \left[\frac{2 - 2\cos(k\Delta x)}{(\Delta x)^2} + \alpha u_i^n \frac{\sin(k\Delta x)}{\Delta x} - \beta\right] \Rightarrow \left|\frac{E_m(t+\Delta t)}{E_m(t)}\right| \le 1 \tag{37}$$

Which implies

$$\left|1+\left[\frac{2-2\cos(k\Delta x)}{(\Delta x)^2}+\alpha u_i^n\frac{\sin(k\Delta x)}{\Delta x}-\beta\right]\right|\le 1 \tag{38}$$

2.5. Convergence Analysis

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\beta u_i^{n+1}(1-u_i^n)-\frac{\alpha u_i^n}{2\Delta x}(u_{i+1}^n-u_{i-1}^n)$$
$$+\frac{1}{(\Delta x)^2}\left[\theta(u_{i+1}^n-2u_i^{n+1}+u_{i-1}^{n+1})+(1-\theta)(u_{i+1}^n-2u_i^n+u_{i-1}^n)\right] \tag{39}$$

The truncation error is given by

$$T_i^{n+1}=\frac{u(x_i,t_{n+1})-u(x_i,t_n)}{\Delta t}-\beta u(x_i,t_{n+1})(1-u(x_i,t_n))$$
$$+\frac{\alpha u(x\ ,t_n)}{2\Delta x}\left(u(x_{i+1},t_n)-u(x_{i-1},t_n)\right)$$
$$+\frac{1}{(\Delta x)^2}\left[\theta(u(x_{i+1},t_{n+1})-2u(x_i,t_{n+1})+u(x_{i+1},t_{n+1}))+(1-\theta)(u(x_{i+1},t_n)-2u(x_i,t_n)+u(x_{i-1},t_n))\right] \tag{40}$$

Here, $t_n=t_0+n\Delta t,\ x_i=x_0+i\Delta x,\ t_{n+1}=t_n+\Delta t,\ x_{i+1}=x_i+\Delta x,\ t_{n-1}=t_n-\Delta t,\ x_{i-1}=x_i-\Delta x$

$$T_i^{n+1}=\frac{u(x_0+i\Delta x,t_n+\Delta t)-u(x_0+i\Delta x,t_0+n\Delta t)}{\Delta t}$$
$$-\beta u(x_0+i\Delta x,t_n+\Delta t)(1-u(x_0+i\Delta x,t_n+\Delta t))$$
$$+\frac{\alpha u(x_0+i\Delta x,t_0+n\Delta t)}{2\Delta x}(u(x_i+\Delta x,t_0+n\Delta t)$$
$$-u(x_i-\Delta x,t_0+n\Delta t))+\frac{1}{(\Delta x)^2}\begin{bmatrix}\theta(u(x_i+\Delta x,t_n+\Delta t)-2u(x_0+i\Delta x,t_n+\Delta t)+u(x_i+\Delta x,t_n+\Delta t))\\+(1-\theta)(u(x_i+\Delta x,t_0+n\Delta t)-2u(x_0+i\Delta x,t_0+n\Delta t)+u(x_i-\Delta x,t_0+n\Delta t)\end{bmatrix} \tag{41}$$

Expanding the right hand side by Taylor Series, we have that the weighted average method is of order two that is, $O((\Delta x)^2,(\Delta t)^2)$

RESULTS AND DISCUSSION

The weighted average method solution of the Burger-Fisher equation is derived as

$$2u_i^{n+1}\left(1-\Delta t\beta+\Delta t\beta u_i^n+r\right)-r\left(u_{i+1}^{n+1}+u_{i-1}^{n+1}\right)+\alpha pu_i^n\left(u_{i+1}^{n+1}-u_{i-1}^{n+1}\right)=2u_i^n\left(1-r\right)+r\left(u_{i+1}^n+u_{i-1}^n\right) \tag{42}$$

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Where $r = \frac{\Delta t}{(\Delta x)^2}$, $p = \frac{\Delta t}{\Delta x}$

For any selected value of α and β.

The method is tested for various parameters and the results are given below

Table 1: Comparison of the weighted average method solution and exact solution of the Burger-Fisher's equation with $\alpha = -1, \beta = 2, \Delta t = 0.01, \Delta x = 0.1, t = 0.1$.

x	Weighted average method	Exact	Error
0.1	0.568180911	0.568319984	0.000139073
0.2	0.580246091	0.580542305	0.000296214
0.3	0.592247089	0.5926666	0.000419511
0.4	0.604203077	0.604679085	0.000476008
0.5	0.616115377	0.616566505	0.000451127
0.6	0.62796343	0.628316188	0.000352758
0.7	0.639706243	0.639916097	0.000209853
0.8	0.651304278	0.651354865	5.05866E-05
0.9	0.662702171	0.662621837	-8.0334E-05

Table 2: Comparison of the weighted average method and exact solution of the Burger-Fisher's equation with $\alpha = 1, \beta = 2, \Delta t = 0.001, \Delta x = 0.1, x = 1, t = 0.1, 0.2, 0.3, 0.4, 0.5$

x	t	Weighted Average method	Exact	Error
1	0.1	0.673707099	0.673707099	0

0.2	0.721115178	0.721115178	0
0.3	0.76404758	0.7640476	1.98E-08
0.4	0.802183888	0.802183889	5E-10
0.5	0.835483527	0.835483537	1.01E-08

Table 3: Comparison of the weighted average method and exact method of the Burger-Fisher's equation with $\alpha = 1, \beta = 2, \Delta t = 0.001, \Delta x = 0.1, x = 2, t = 0.1, 0.2, 0.3, 0.4, 0.5$

x	t	Weighted Average Method	Exact	Error
2	0.1	0.772941659	0.772942259	0.0000006
	0.2	0.8099983	0.809998434	0.000000134
	0.3	0.842241311	0.842241313	0.000000002
	0.4	0.869891523	0.869891526	2.5E-09
	0.5	0.893309405	0.893309406	9E-10

Table 4: Comparison of the weighted average method and exact method of the Burger-Fisher's equation with $\alpha = 1, \beta = 2, \Delta t = 0.001, \Delta x = 0.1, x = 3, t = 0.1, 0.2, 0.3, 0.4, 0.5$

x	t	Weighted Average Method	Exact	Error
3	0.1	0.848771745	0.848771748	2.3E-09
	0.2	0.87544664	0.875446642	1.9E-09
	0.3	0.897981931	0.897981931	3E-10
	0.4	0.916827302	0.916827304	1.7E-09
	0.5	0.932453258	0.932453309	5.06E-08

3.1. Graphical representation of results

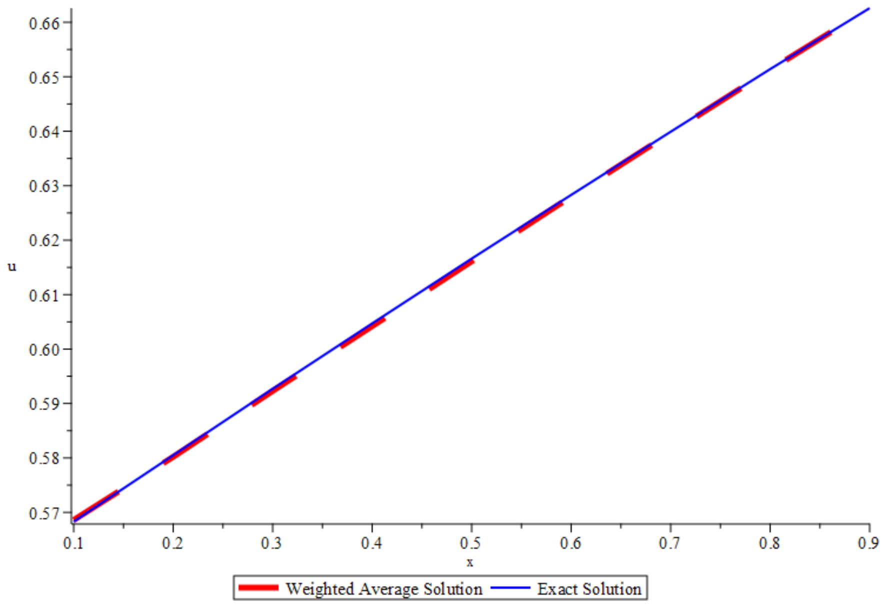

Figure 1: Comparison of the weighted average method solution and exact solution of the Burger-Fisher's equation with $\alpha = -1, \beta = 2, \Delta t = 0.01, \Delta x = 0.1, t = 0.1$

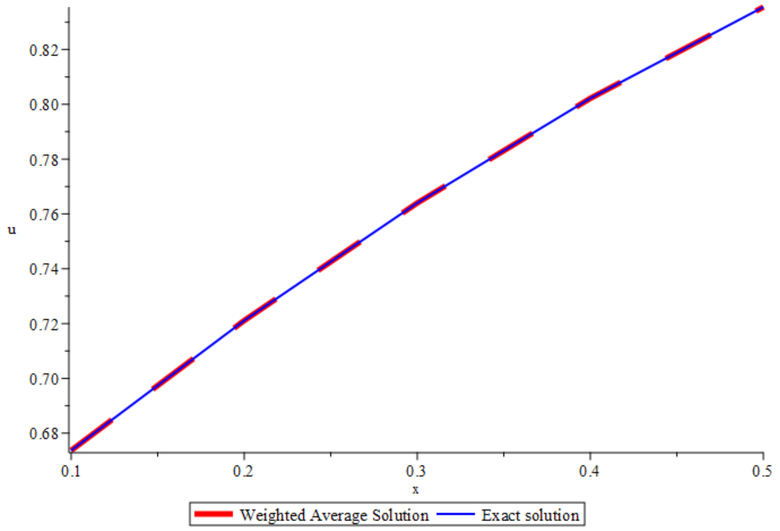

Figure 2: Comparison of the weighted average method and exact solution of the Burger-Fisher's equation with $\alpha = 1, \beta = 2, \Delta t = 0.001, \Delta x = 0.1, x = 1, t = 0.1, 0.2, 0.3, 0.4, 0.5$

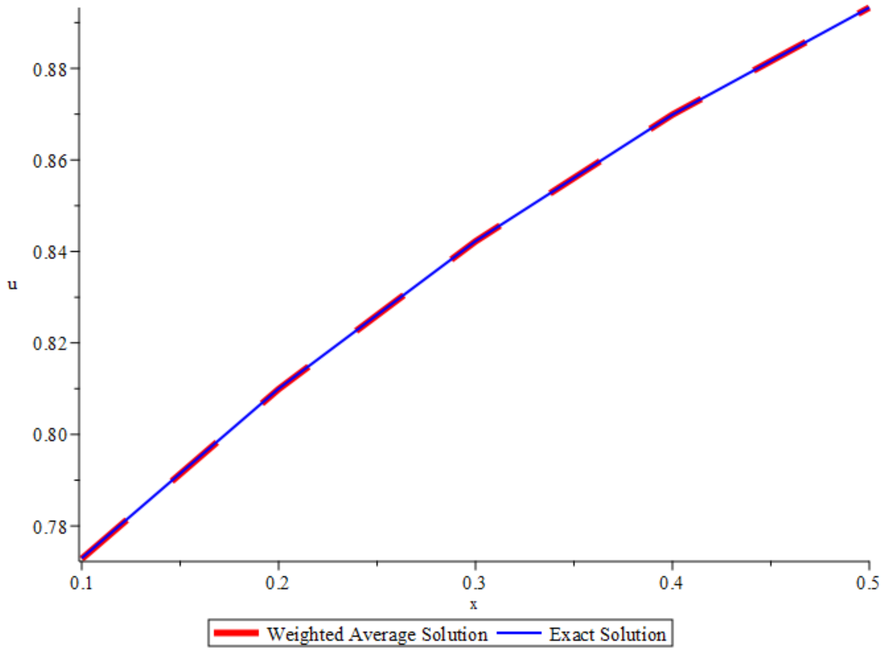

Figure 3: Comparison of the weighted average method and exact method of the Burger-Fisher's equation with $\alpha = 1, \beta = 2, \Delta t = 0.001, \Delta x = 0.1, x = 2, t = 0.1, 0.2, 0.3, 0.4, 0.5$

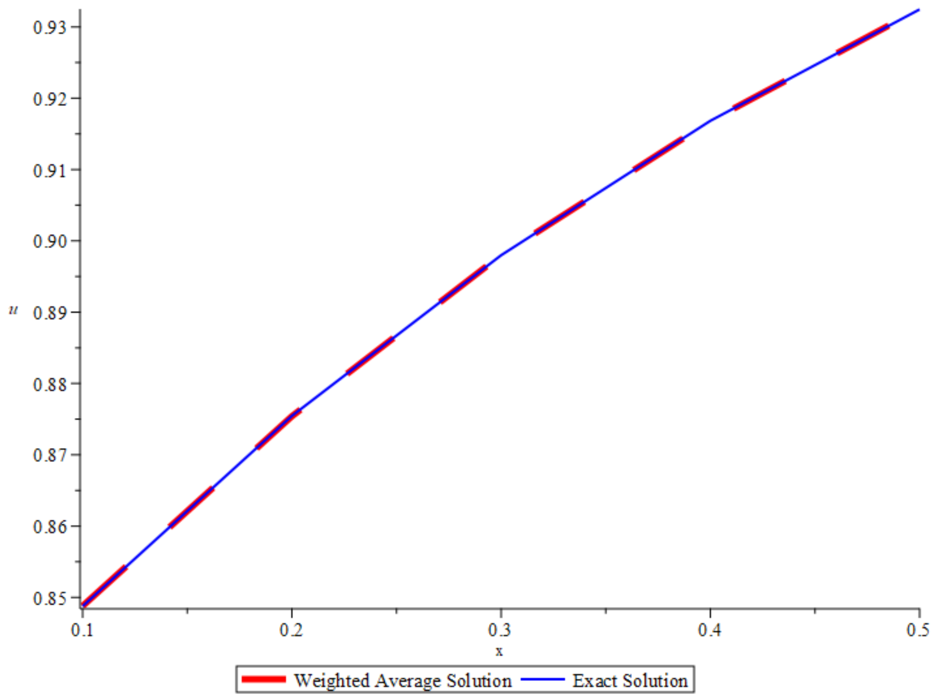

Figure 4: Comparison of the weighted average method and exact method of the Burger-Fisher's equation with $\alpha = 1, \beta = 2, \Delta t = 0.001, \Delta x = 0.1, x = 3, t = 0.1, 0.2, 0.3, 0.4, 0.5$

DISCUSSION OF RESULTS

The results presented in Tables 1 and 2, along with the graphical representations in Figures 1, 2, and 3, provide valuable insights into the performance of the weighted average method for solving the Burger-Fisher equation.

Comparison of Numerical and Exact Solutions

- In Table 1, the comparison between the numerical solutions obtained using the weighted average method and the exact solutions at different spatial locations (x) for a fixed time (t = 0.1) reveals small errors. Similarly, Table 2 compares the solutions at different times (t) for fixed spatial locations (x = 1, 2, 3).

- The observed errors between the numerical and exact solutions are negligible, indicating the accuracy of the weighted average method in approximating the solutions to the Burger-Fisher equation.

Implications for Method Accuracy

- The small errors observed suggest that the accuracy of the weighted average method is high when applied to the Burger-Fisher equation. This finding is crucial for ensuring the reliability of numerical solutions obtained through this method.

- The implication that smaller changes in the time level ($\Delta t \rightarrow 0$) lead to higher accuracy underscores the importance of controlling numerical parameters, such as time step size, to achieve precise results.

Graphical Representation

- Figures 1, 2, and 3 visually depict the comparison between the numerical and exact solutions, with the red crosses representing the solutions obtained using the weighted average method and the blue circles representing the exact solutions.

- The close proximity of the data points in the graphs further illustrates the small errors between the numerical and exact solutions, reinforcing the accuracy of the weighted average method.

Practical Considerations
- The findings highlight the practical utility of the weighted average method for solving the Burger-Fisher equation with high accuracy. Researchers and practitioners can rely on this method to obtain reliable numerical solutions for various applications in fields such as fluid dynamics, population dynamics, and reaction-diffusion systems.

- The results also emphasize the importance of fine-tuning numerical parameters, particularly the time step size, to ensure optimal accuracy when using the weighted average method.

SUMMARY

The study utilized the weighted average method to tackle the Burger-Fisher equation, a nonlinear partial differential equation (PDE). Employing both explicit and implicit discretization techniques, the method was applied to approximate solutions. To gauge its efficacy, the study conducted convergence and stability analyses. Notably, the weighted average method demanded significant computational resources, necessitating the utilization of mathematical software like MATLAB and MAPLE. At each time step, the method yielded a tridiagonal matrix system, which was solved using MAPLE17. The study compared the solutions obtained via the weighted average method with exact solutions, enabling a comprehensive assessment of its accuracy. A line graph was generated to visually represent the relationship between the numerical and exact solutions at different time steps. Overall, the study demonstrated the practical applicability of the weighted average method in solving the Burger-Fisher equation, shedding light on its computational requirements, convergence properties, and stability characteristics.

CONCLUSION

In conclusion, this study highlights the effectiveness of the weighted average method in addressing nonlinear partial differential equations, with a specific focus on the Burger-Fisher equation. Through rigorous analysis, it was determined that the method exhibits unconditional stability and second-order accuracy when applied to this equation. Comparison between numerical solutions obtained using the weighted average method and exact solutions revealed a high level of agreement, with negligible errors observed. This indicates the method's accuracy and reliability in approximating solutions to the Burger-Fisher equation. Overall, the findings underscore the suitability of the weighted average method as a viable approach for solving nonlinear partial differential equations, offering researchers a valuable tool for addressing complex mathematical problems in various fields of study.

Competing interest
The authors declare that they have no competing interest.

Data Availability
The datasets generated during and/or analyzed during the current study are available from the corresponding author on request.

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