STUDENTS' VISUAL REASONING OF THE CONNECTION BETWEEN FUNCTION AND ITS DERIVATIVE : A GRAPHICAL APPROACH

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Abstrak

Kajian artikel adalah sebahagian dapatan daripada projek di mana pembangunan penaakulan visual pelajar diterokai semasa menyelesaikan masalah pengkerbedaan berasaskan grafik. Kesulitan pelajar dalam memahami konsep pengkerbedaan and seterusnya mengaitkannya dengan fungsi asal telah dikenalpasti. 194 pelajar pra-universiti telah diberi tugasan untuk mengukur kebolehan mereka menghubungkaitkan graf pengkerbedaan dengan fungsinya dan cara penyelesaian masalah dianalisa. Dapatan menunjukkan bahawa pelajar mampu menggunakan kedua-dua kaedah grafik (gambarajah tanda dan melakar graf fungsi asal) dan kaedah manipulasi algebra. Implikasi pembelajaran juga dibincangkan.

Kata kunci *Pendidikan matematik, penaakulan visual, graf, pengkerbedaan*

Abstract

The study described in this article is part of a larger project, in which we track the development of students' visual reasoning when solving derivative problems using graphs. Students' difficulties in understanding derivatives and further to relate to their original functions have been well acknowledged. We engaged 194 pre-university students with tasks measuring their ability to connect graphs of derivative and properties of functions and analyzed their solution methods. Finding shows that students employed both graphical (using sign diagram and sketching the original function) and algebraic methods. Implication of instructions are discussed.

Keywords *Mathematics education, visual reasoning, graphs, derivative.*

INTRODUCTION

Researchers, educators and mathematicians internationally have advocated that one of the fundamental goal of mathematics education, at all levels, should be the development of reasoning skills (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). At the same time, calculus is a mathematical topic taught at secondary levels and with vast applications at the higher levels of education. Therefore, Malaysia, in abiding to the effort made by the United States and Europe, calls for some adjustment in the calculus instructional methods. As a result, the Higer Education Ministry has directed for steps to be taken to ensure that the proposed changes in the curriculum and the ways materials are delivered will answer these calls (Ball et al., 2002) as to promote the comprehension and ability to communicate mathematical understanding (Hanna & Jahnke, 1996; Polya, 1981; Stylanides, 2007). One elemental variation is the use of graphs in fostering the visual reasoning skills in students, as Zimmerman (1991) asserted : "Conceptually, the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (p.136).

The ability to reason visually is imperative for students to develop and appreciate convincing arguments (Baker, Cooley & Trigueros, 2000; Ubuz, 2007). However, researchers and educators have noted that many difficulties inhibit visual reasoning from being practiced and emphasized in the classrooms (Ball et al., 2002). For example, although mathematics curriculum outlined the use of technology such as graphics calculators, many teachers still emphasize the learning of algorithms and procedures in the secondary and pre-university levels. Students are indirectly trained to disconnect content from its underlying concepts (Kamii & Dominck, 1998). In addition, many students view representations such as graphs, as a procedure or as a series of steps to undertake without engaging logical reasoning or justification for their ideas or solution steps (Kamii & Dominck, 1998). Consequently, they proceed to the higher level of education not fully-equipped with reasoning power.

Concepts of functions and derivatives are central to the study of calculus, but many difficulties were reported when students need to understand them. Further, little is known about how students acquire the ability to graphically relate the connection between the two. This paper reports results from investigating pre-university students' understanding the connection between the properties of a function and its derivatives. It is part of longer study trying to investigate the development of visual reasoning when solving derivative problems using Cartesian graphs. Visual reasoning has been shown to be an important ability for percepting the behaviour of functions and interpreting related derivative properties. Therefore the study aimed to seek solution for the following query : What perceptual and cognitive processes are involved in the process of extracting properties of function from the graphs of its derivative?

THEORETICAL FRAMEWORK AND RELATED LITERATURE

Student understanding of derivative

Differential calculus is mostly taught according to textbook approach and through rote learning. Students were presented with sequences of procedures and steps in order to differentiate various types of functions (Chappel & Kilpatrik, 2003). Most of the students have procedural and algebraic understanding of the relationships between functions and their derivatives (Lithner, 2000). Graphing of either functions or their derivatives are also trained the same way. Stahley (2011), in his thesis, splited the students' understanding of derivative into two types : the first is the 'pointwise' understanding where the students are able to recognize the types of turning points i.e. the local maxima, local minima or stationary inflexion, and the second as the 'across-time' understanding where the students are able to realize the graphical relationship between a function and its derivative (Monk, 1994). Various researches were conducted with the purpose to investigate the students' abilities in understanding functions and their derivatives both algebraically and graphically (Asiala, Cottril, Dubinsky& Swingendorf, 1997; Aspinwall, Shaw, & Presmeg, 1997; Borgen & Manu, 2002; Tiwari, 2007; Ubuz, 2007; Zandieh, 2000). Among the students' definition of derivative from Tiwari (2007) study was that derivative is used to show the relative maximum or minimum points and the intervals when the functions are increasing or decreasing.

Information extraction

After Orton's (1983) "It is known that some students are introduced to differentiation as a rule to be applied without much attempt to reveal the reasons for and justifications of the procedure. Many regard this as bad educational practice, and, in fact, it should not be necessary" (p.242), the last four decades had seen an encouragement in mathematics educational research emphasizing on understanding the concepts and the graphing aspects of calculus. Theories on graph comprehension are mostly focussed on the conceptual and perceptual processes specifically on the extraction of information from graphs (Ratwani, 2008). Lohse's (1993) task analytic theory describes the process of interpreting information from basic graphs such as Cartesian or line graphs with two dimensional coordinate system relating magnitudes of two (or more, in case of parametric functions) quantities. Starting with 'pattern recognition', students examine the graphs to look for visual information as dictated in the questions (Peeble & Cheng, 2003). They may need to go back and forth from the axes and the line or curve representing the quantities in problem searching for specific information. Encoding process might also take place in some situations (Lowrie, 2010). They will then determine the 'conceptual relation' to make meaning of the visual information quantitatively. Finally, the visual information are relate back to the graph to establish the perceptual-conceptual relationships. Lohse's model was founded on the assumption that the process of decoding from graphs portrayed the students' knowledge on graphs properties and characteristics.

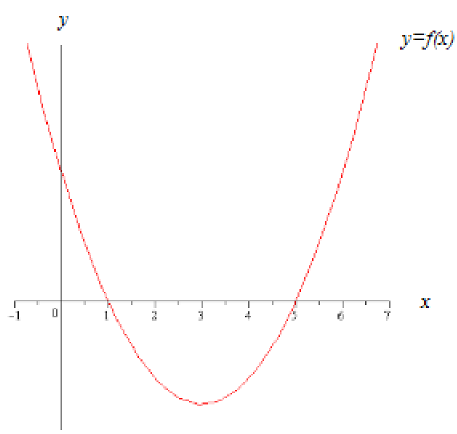
This outlook offers explanation for the most common difficulties that students face and consequently performing errors when extracting properties of function from their derivative graphs. Studies that had investigated students working with graphs of functions and their derivatives revealed that students tend to confuse themselves between the slope of the function, or the slope of the tangent to the curve of the function and the slope of the derivative function. Summarizing, they were unable to coordinate and relate the two different attributes of the function. A visual connection of the two quantities is designed in the form of graph as task for the students.

METHODOLOGY

The framework was used to design the tasks for the students. The validity of the tasks item was confirmed by two experts in the area while the interrater reliability was calculated to be 0.87. The tasks were distributed to 194 pre-university students studying the South Australian Matriculation programme in Malaysia with the intention to pursue to various disciplines at tertiary level. At the time of the study, they had finished the calculus syllabus. Prior to pre-university curriculum, the students were also exposed to the concepts of derivative for at least a year period at the secondary level. Their worked solutions were analyzed looking for the methods they employed in dealing with graphs provided. Students were given no specific instruction on the method(s) that they can employ. They, therefore may opt to : 1) using the sign diagram of the derivative function and look for intervals where the signs change from negative to positive vice-versa, 2) sketch the graph of the original function and determine the properties of the graph, and 3) using the algebraic manipulation by the process of differentiation to find the second derivative. Due to space limitation, only one task will be discussed in this article.

Task :

The diagram shows that graph of the derivative function of the curve $y = f(x)$. *For what value of x does* $f(x)$ *have a local minimum? Justify your answer. Outline all steps taken to arrive to the answer.*



The purpose of the task was to investigate if the students are able to see the relationship of the function and its derivative graphically. The given graph of the derivative function cuts the *x*-axis at the point *x* = *1* and *x* = *5* and students should realize that these are the turning points since $f''(x) = 0$. They should further analyze the positivity or negativity of the curve through the location of the graphs (above or below the *x-axis* respectively) before and after the points to determine the nature (local maxima or local minima) of the points.

FINDINGS

In this paper, we present our overall findings from comparing the students' method of arriving to the solution and then describe the students' activity during the task that illustrates the variety of visual reasoning used as justifications for solutions. Due to space limitation, the part '*Outline all steps taken to arrive to the answer'* is not presented in this article.

Classifying solutions

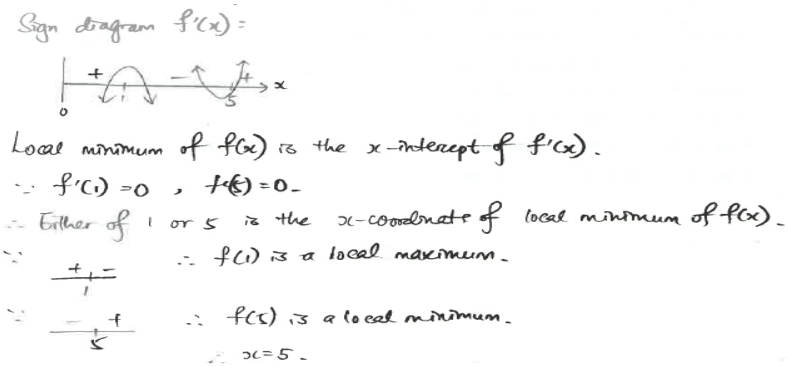
The students worked solutions were classified by the approaches they employed as shown in Table 1. Our analysis of the visual reasoning performed during this task centered on the identification and description of the drawing of sign diagrams, sketching of the graph of function and reverting to algebraic manipulations. It can be seen that more than half, 63(58.76%) of the students opted to visual approach; with 63(32.47%) drew the sign diagram and only 8(4.12%) of them made some mistake interpreting the diagram. Another 51(26.29%) of the students decided to sketch the graph of the function and worked the solution from there with a higher portion of 29(14.95%) performing errors when dealing with the graphs.

Table 1 The frequencies and percentages for each methods employed by the students

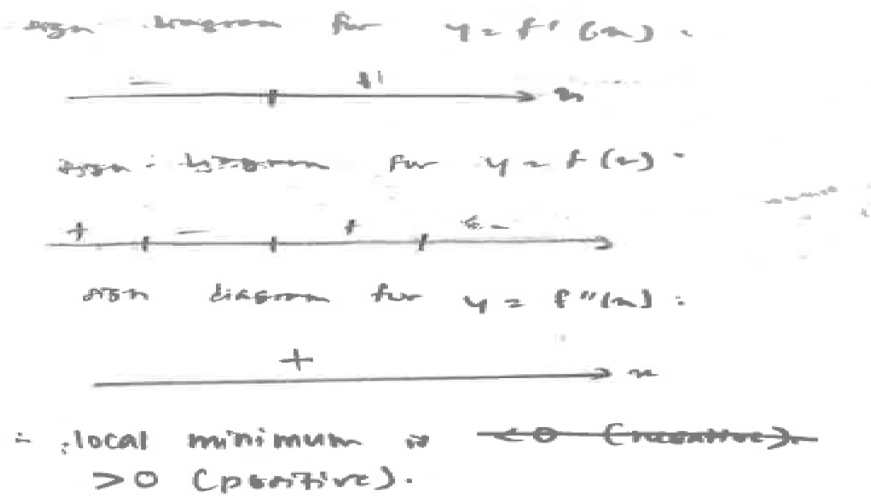
Approach		Frequency	(Percentage)
Sign diagram	Correct	55	(28.35)
	Incorrect	8	(4.12)
Graph of function	Correct	22	(11.34)
	Incorrect	29	(14.95)
Algebraic manipulation	Correct	71	(36.60)
	Incorrect	4	(2.06)
Not attempted		5	(2.58)
Total		194	(100.00)

The next section highlighted the samples of students' worked solutions for each approach.

Drawing sign diagram



(a) Correct solutions

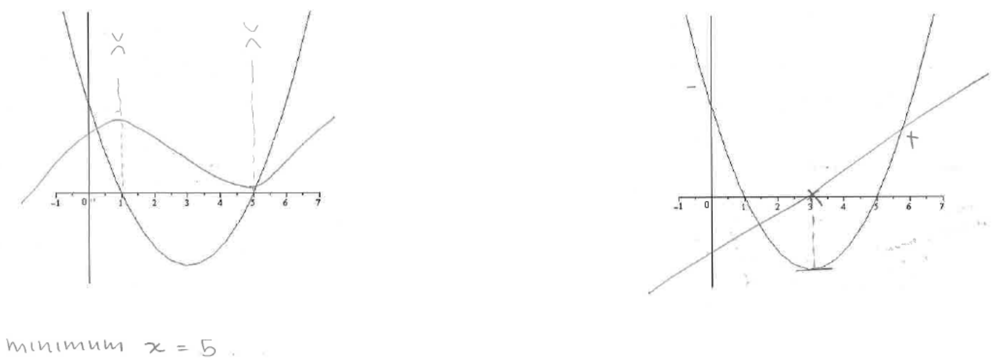


(b) Incorrect solution

Figure 1 Samples of students with drawing sign diagram

In the correct solution as illustrated in Figure 1(a), student managed to ‘convert’ the graph of the derivative function into a sign diagram – the diagram showing the intervals and signs of the derivative function throughout the domain. The student then encode the positive and negative signs of the derivative function for the increasing and decreasing intervals of the function (the curves with arrows in the sign diagram) and hence finalizing the local minima of the function. In Figure 1(b) the student confused him/herself among the sign diagrams of all three functions; the main or original function, the derivative function and the second derivative function. This particular student seemed to have weak ideas on the concepts of derivative. This can be seen when he/she did not even assigned the *x*-coordinates on the horizontal axis of the diagrams and consequently, although concluded the right concept – a minimum occur when the second derivative is positive (through pattern recognition or memorization perhaps), no final answer was presented.

Graph of function

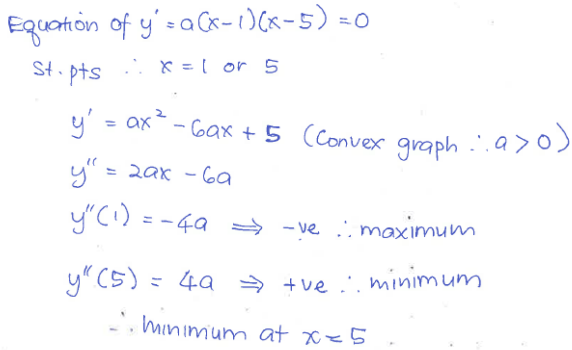


(a) Correct solutions (b) Incorrect solution

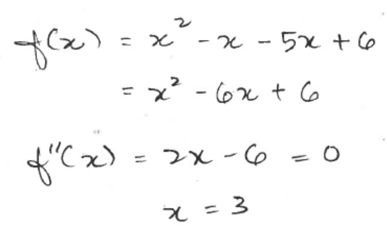
Figure 2 : Samples of students with sketching graph of function

In the correct solution or correct graph sketched by the student, as illustrated in Figure 2(a), he/she managed to realize that the function has a turning points at points $x = 1$ and $x = 5$ by (assuming) first indicating the $\cup \cap$ signs followed by the sketching of the graph of the function based on the location of the derivative graph indicating the increasing or decreasing of the function. Figure 2(b) shows that the student either did not read or did not understand the question properly or had avery shallow knowledge on the concepts of derivative. He/she presented the basic idea or definition of (local) minima and read of the answer from the graph. His/her sketched of straight line graph which represented the derivative function was actually of no assistance.

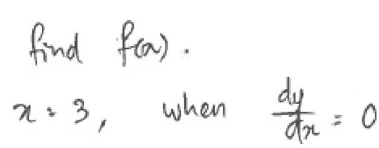
Algebraic manipulation



(a) Correct solutions



(i)



(ii)

(b) Incorrect solutions

Figure 3 : Samples of students with algebraic manipulations

In the correct solution as illustrated in Figure 3(a), the student confirmed the reluctance to use the graph to arrive to the solution. Nevertheless, he/she was able to extract data and applied relevant concepts of differentiation calculus to set up algebraic equations. On the other hand, the worked solutions in Figure 3(b) displayed the algebraic method of totally wrong concepts together with an arithmetic error performed. In Figure 3(b)(i) the student had in mind that basically, finding any stationary point would require the differentiation of functions and equating it to zero. In Figure 3(b)(ii) the student was totally confused of the concept or may have been innorance in reading the question. He/she had assumed that graph given will always represent the graph of functions and read-off the x-coordinate of the point. In this case he managed to relate the minimum point of the graph witht he basic property of teh derivative function, dy/dx = 0.

CONCLUSION

Pre-university students have always portrayed calculus, specifically differential calculus as one of the most difficult topics in mathematics. They often misunderstood the notion of derivative (and equally on the *rate of change*) especially when presented in graphical forms. Most of the students are able to follow the algebraic rules of differentiation, be it the chain rule, product rule, quotient rule, etc. The visual approach seems to be very difficult to most students (Uesaka, Manalo, 2011). Since the setting to extract and explain concepts posed in the task was non-traditional, students exhibited some struggles and resistance to the new approach.

As supported by Chappell and Kilpatrik (2003), it is clear from the findings of the study that some students need to broaden the previous and existing knowledge when encountering unfamiliar situations. Those with strong conceptual understanding tend to complete the task with minimum confusion. Also as agreed by Lithner's (2000) findings, some of the students may be able to blindly use the formula or procedural method to solve any derivative task without grasping the concepts and implications of how and why they work (Stahley, 2011).

To summarise, it is clear from the results of the study that some of the students demonstrate rote memorization and procedural understanding of functions and their derivatives. Therefore, they are likely in need to expose to alternative methods of delivering the concepts and classroom instructional. Much attention should be focussed to foster student's ability on the use of graphs as visual reasoning tools. If this is not implemented, students are likely to be progressively complacent with their procedural method of solution (especially those who manage to arrive to the correct solution) and may not tolerate in their mathematical problem solving experiences at the higher level of educations. The study is hoped to expand on the literature of students' difficulties and conceptual understanding of functions and their derivatives and to enhance our awareness of students' thought processes and visual reasoning skills.

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