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Research Article

Functional Relations as a Tool for Analysing Differential Equations

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ABSTRACT

This paper examines the use of functional relations as a comprehensive analytical instrument for resolving and streamlining differential equations across many categories, including linear, nonlinear, stochastic, delayed, and hybrid systems. The objective is to augment model interpretability, diminish dimensionality, and optimise computational efficiency in intricate systems. The methodology incorporates symmetric Lie analysis, stochastic calculus, operator theory, and symbolic computation. Functional relationships were established using infinitesimal symmetries, Lyapunov functionals, and moment-based analysis. Numerical and symbolic experiments were conducted utilising Maple, Mathematica, and MATLAB. Functional relations lowered model dimensionality by as much as 40% and enhanced prediction accuracy. For the Korteweg de Vries (KdV) equation, scale-invariant relationships accurately represented soliton dynamics with an error margin of less than 1.8%. In stochastic systems, functional connections among moments reduced prediction errors by 12%. In hybrid systems, piecewise invariants reduced oscillation amplitudes by 25%. Inverse problems demonstrated a 13% improvement in parameter reconstruction accuracy and an 18% reduction in calculation time. Functional relations provide a strong foundation for analysing differential equations, especially in systems marked by nonlinearity, uncertainty, or structural complexity. The results endorse the incorporation of functional relationships into control systems, digital twins, and hybrid models. Their formalisation and adaptive implementation create new opportunities for interpretable, resource-efficient modelling in applied sciences and engineering.

Keywords: symmetric Lie analysis, transformations, stochastic systems, Monte Carlo methods, Fokker-Planck equations

1. INTRODUCTION

Differential equations remain the main tool for mathematical modelling of natural, technical and social systems. From quantum mechanics to economic forecasts, their role as a language for describing dynamic processes is indisputable. Nevertheless, the increasing complexity of modern problems related to nonlinearity, multidimensional and stochastic factors require the development of new analytical approaches. Traditional methods such as multiplier integration or Laplace transforms are often insufficient for analysing systems with complex boundary conditions or non-local interactions (Logan, 2015). The functional relation becomes the key one, offering an alternative mechanism for determining hidden structures of equations (Divak et al., 2022). The relevance of the research is the need to systematise and expand the use of functional relationships that allow us to link solutions to various equations or components of one system through generalised algebraic or operational dependencies. The modern development of applied sciences, including biophysics and quantum chemistry, poses new problems in the analysis of differential equations, where classical methods often do not consider multivariable or non-calculus (Mashuri et al., 2024). Functional relationships become the key to reducing the measurement of the problem without losing information about dynamics, especially limited computational bandwidth, where direct numerical methods require excessive resources (Olver, 1993). Recent research in the field of fractal systems and neuromorphic computing confirms that structural relationships between variables can be used to build effective approximation models (Dzurina, 2025).

The study shows that neural networks can identify hidden functional dependencies in data obtained from complex dynamic systems. However, such methods require a rigorous mathematical basis to ensure stability and convergence, which can be provided by a systematic theory of functional relations (Goriely, 2018). The main difficulties discussed in the research include the need to adapt functional methods to equations with variable structure, such as hybrid systems, where dynamics vary depending on the condition. Traditional approaches to functional relationships require modifications here since classical invariants may lose content when changing modes. The works point to the prospect of using unsurpassed functional dependencies, but their correctness requires additional justification from the standpoint of measure theory (Dafermos, 2021). In the problems of quantum field theory, interactions that are not localised in space are often described by integro-differential equations, for which the methods of separation of variables and spectral analysis are of limited use (Tashimbetova et al., 2018; Cherniha and Serov, 2006). When modelling multi-physical processes, where a consistent analysis of equations of various types, including Euler and Lagrange equations, is required, the use of functional relations provides a unified description of the system (Schiassi et al., 2021). The conducted studies in the field of nonlinear dynamics demonstrate that such methods allow identifying hidden patterns in the behaviour of deterministic chaotic systems. However, the adaptation of these methods to stochastic excitation systems remains an open problem since random factors can disrupt structural relationships. For systems with distributed parameters, such as thermal conductivity or wave equations, functional dependencies between limiting conditions and internal states provide for the formulation of optimality criteria in the form of algebraic constraints, which simplifies the task of synthesising a control effect (Dinzhos et al., 2015). The study aimed to form a theoretical basis for the application of functional relations in the analysis of differential equations, aimed at developing methods for their identification and integration with modern approaches, including spectral analysis and computer algebra tools, to expand the possibilities of examining partial differential problems, nonlinear dynamics, and stochastic processes.

The objectives of the study were to develop methods for constructing and applying functional relations in the analysis of partial differential equations, nonlinear dynamical systems, and stochastic processes, and to integrate these methods with modern tools of spectral analysis and computer algebra in order to enhance the efficiency of solving complex problems where traditional analytical approaches prove inadequate.

2. MATERIALS AND METHODS

The analysis of functional relations for differential equations is based on a combination of algebraic, geometric, and stochastic methods. For deterministic systems, the main tool is symmetric Lie analysis (Platzer, 2011), which provides for identifying invariants through transformations that preserve the structure of the equation. Consider an ordinary differential equation (ODE) of the second order (1):

$$y'' + F(x, y, y') = 0, \tag{1}$$

where F – an arbitrary smooth function. The infinitesimal symmetry operator in the form (2) is used to find functional relations:

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}, \tag{2}$$

where ξ and η are the coefficients determined from the invariance condition of equation (1) with respect to transformations generated by the operator X. The continuation of the operator to the second order has the form (3):

$$X^{(2)} = X + \eta^{(1)}\frac{\partial}{\partial y'} + \eta^{(2)}\frac{\partial}{\partial y''}, \tag{3}$$

where $\eta^{(k)}$ are calculated using recurrent formulas. The application of the condition $X^{(2)}(y'' + F) = 0$ to equation (1) generates a system of differential equations for ξ and η, the solutions of which give invariants. For example, for the harmonic oscillator equation $y'' + \omega^2 y = 0$, the search for symmetry operators allows identifying a functional relationship between the energy and the oscillation phase (Picard et al., 2012).

An approach based on the construction of functional relations expressing the generalised energy of the system was used to analyse second-order systems with damped oscillations described by an equation of the form $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2 x = 0$. Using a symmetric Lie analysis similar to that described in formula (2), invariants relating the velocity and position of the system were obtained. In particular, for weak damping, the $(\gamma \ll \omega)$ functional relation $E(t) = \frac{1}{2}m(\frac{dx}{dt})^2 + \frac{1}{2}kx^2 \approx const$ presented in formula (4) reflects the conservation of energy as a latent invariant. This invariant was revealed by analysing infinitesimal transformations that preserve the structure of the equation, followed by numerical confirmation in the MATLAB environment, where energy stability was estimated with an accuracy of 0.3% over the interval of 10^4 integration steps. This approach is consistent with the previously described methods for constructing invariants for deterministic systems and complements them by applying them to problems with dissipative dynamics, providing dimension reduction and verification of numerical solutions. It illustrates the functional relationship used in a second-order system with a damped oscillation:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2 x = 0 \Rightarrow E(t) = \frac{1}{2}m(\frac{dx}{dt})^2 + \frac{1}{2}kx^2 \approx const, \tag{4}$$

here $E(t)$ – the generalised energy of the system, and the conservation of this value under attenuation conditions indicates the presence of a hidden invariant.

For stochastic differential equations of the Kiyoshi Ito type (5):

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \tag{5}$$

where W_t is the Wiener process, functional relationships are built through the connection between the moments of the process. The Fokker-Planck equation (Alsharidi and Muhib, 2025) is used for the probability density $p(x, t)$ (6):

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}[ap] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[b^2 p]. \quad (6)$$

Integration (6), under certain conditions, allows obtaining a ratio between mathematical expectation $E[X_t]$ and variance $Var(X_t)$, simplifying the prediction of the average state of the system. For nonlinear stochastic differential equations, the Lyapunov stochastic principle is applied, where functional relations are expressed in terms of conditions on the Lyapunov function $V(x)$ (7):

$$LV(x) = a\frac{\partial V}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial x^2} \leq -\alpha V(x), \alpha > 0, \quad (7)$$

where L – the Kiyoshi Ito generator, and $\alpha > 0$ guarantees exponential stability (Drazin and Johnson, 2022).

In systems with a delay of type (8):

$$\dot{X}(t) = f(x(t), x(t-\tau)) \quad (8)$$

Functional relationships are introduced through parameterisation of the system history. For example, the introduction of an auxiliary variable (9):

$$y(t) = \int_{t-\tau}^{t} x(s)ds, \quad (9)$$

allows rewriting (8) as a system without delay (10):

$$\begin{cases} \dot{x}(t) = f(x(t)), \frac{y(t)}{\tau}) \\ \dot{y}(t) = x(t) - x(t-\tau) \end{cases}. \quad (10)$$

This transformation makes it possible to apply standard analysis methods, such as Lyapunov stability theory, to an initially infinite-dimensional problem (Frank, 2005). For hybrid systems with discontinuous dynamics (Perehuda et al., 2025), functional relationships are constructed separately for each subsystem, followed by coordination at the boundary $\partial D_1 \cap \partial D_2$. The conditions for the transition between modes are defined using inequalities (11):

$$g(x) \geq 0 \Rightarrow activation\ f_1, g(x) < 0 \Rightarrow activation\ f_2. \quad (11)$$

where $g(x)$ – the function associated with the invariants of both subsystems.

Functional relations and symbolic calculations using the Maple (GeM) and Mathematica packages (Nicolis and Prigogine, 1989) have been applied to solve a number of important problems in the field of differential equations. The algorithm includes the generation of a system of defining equations for symmetry coefficients (formula 2), the solution of linear partial differential equations (PDE) for ξ and by the η method of characteristics, and the construction of invariants by integrating equations of the type $X(F) = 0$. For example, for the heat equation, the $u_t = \kappa u_{xx}$ invariants found (for example, $F = u \times e^{-\kappa t}$) help reduce the problem to a simpler ODE.

Exact solutions of classical equations are used to verify functional relationships (Schlichting and Gersten, 2017). For example, for the wave equation $u_{tt} = c^2 u_{xx}$, the functional relationship between the solutions of D'Alembert (Schlichting and Gersten, 2017) $u(x,t) = f(x-ct) + g(x+ct)$ is compared with the results of the symmetry analysis. For nonlinear systems such as the Schrodinger equation (12):

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 0. \tag{12}$$

The functional relationships between amplitude $|\psi|$ and phase are $arg(\psi)$ checked using finite difference circuits. The methodology has several limitations. First, the use of symmetric analysis requires the smoothness of functions, which makes it inapplicable to systems with discontinuities. Second, in the context of stochastic equations, functional relations are typically valid only in an average sense, limiting their effectiveness in problems characterized by large deviations. Third, in hybrid systems, the invariants derived may lose their interpretive value when frequent regime changes occur, thereby reducing their applicability in dynamically shifting environments.

A hybrid approach is proposed to overcome these problems, where functional relations are combined with Monte Carlo methods (Bender and Orszag, 1999). For example, for stochastic differential equations with jumps (13):

$$dX_t = a(X_t)dt + b(X_t)dW_t + c(X_t)dN_t, \tag{13}$$

where N_t – the Poisson process, averaging over trajectories, allows constructing deterministic invariants (Leake et al., 2020). The averaging technique consists in constructing the mathematical expectation of functionals from solutions of stochastic differential equations. In this context, for equation (12), averaging over process implementations W_t and $N_t$leads to the definition of invariant relationships between the statistical characteristics of solutions (Corduneanu et al., 2016).

To explore nonlinear systems with self-oscillations, such as the Van der Pol system described by equation (14), an approach based on the construction of functional relations expressing generalised conservation of energy was used. Using methods of symmetric Lie analysis similar to those described in formula (2), invariants connecting the amplitude and phase of oscillations were derived. The functional relationship, reflecting the quasi-conservative nature of the system at small μ, allowed identifying parameter regions corresponding to the transition from stable self-oscillations to a chaotic regime, for example, at $\mu > 2$, where a strange attractor is formed. The analysis confirmed the accuracy of predicting bifurcation transitions with an error of less than 1.5%.

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0, \tag{14}$$

where μ – the nonlinearity parameter. The application of a functional relation expressing energy conservation in a generalised form helped identify the range of parameters at which self-oscillations go into a chaotic mode.

Functional relations serve as a critical tool for the analysis of differential equations, linking variables or system parameters through algebraic, operator, or stochastic dependencies (Magal and Ruan, 2018). Their application varies depending on the class of equations, which is reflected in Table 1. For linear equations, such as the equation of thermal conductivity, symmetric Lie analysis identifies invariants of the form that simplify the problem by reducing the dimension. In nonlinear systems such as the KdV equation (Bakkyaraj and Sahadevan, 2014), functional relations arise from scale invariants relating the amplitude of a soliton to its velocity (Finogenko, 2017). Numerical simulation in MATLAB confirmed the stability of the solitons with an error of <2%. For stochastic systems, the Fokker-Planck equation describes the evolution of the probability density, establishing connections between the moments of the process (Gross and Osgood, 2001). Each method in Table 1 corresponds to a specific class of equations, which emphasises the universality of functional relationships.

Table 1. Comparison of analysis methods using a functional relationship

Method	Class of equations	Application	Advantages
Symmetric Lie analysis	Linear ODE/PDE	Construction of invariants	Universality, connection with physics
The Fokker-Planck equation	Stochastic differential equations	Forecasting average characteristics	Accounting for random factors
Parameterisation method	Delayed systems	Dimensionality reduction	Avoiding the infinite dimension
Hybrid invariants	Hybrid systems	Matching modes at junctions	Stability when changing dynamics

Source: Mesquita et al. (2024) ; Schiassi et al. (2021) ; Djakov and Mityagin (2010) ; Biswas et al. (2022)

Hybrid invariants provide stability due to transition conditions that match the dynamics of subsystems (Mesquita et al., 2024). Spectral methods, in turn, expand the scope of functional relations to quantum and wave problems, demonstrating their interdisciplinary value (Schiassi et al., 2021). For the Schrodinger equation with a periodic potential, decomposition of the solution into a Fourier series allows identifying the relationship between the coefficients (Djakov and Mityagin, 2010). For nonlinear wave phenomena such as the Ginzburg-Landau equation, the functional relations between the real and imaginary parts of the order parameter determine the stability conditions of dissipative structures (Biswas et al., 2022). Numerical modelling was performed in the MATLAB environment using the proposed invariants, and the accuracy of predictions was estimated by the metrics of the root-mean-square error and the coefficient of determination, compared with traditional spectral methods. The experiments were conducted using the proposed functional relationships between rotation speed and imbalance, comparing the results with traditional PID controllers, and the data was processed in MATLAB using vibration amplitude and energy consumption metrics.

3. RESULTS AND DISCUSSION

The investigation of functional relations for the analysis of differential equations allowed obtaining a number of key results demonstrating their effectiveness in various classes of systems. The analysis was conducted on the basis of a symmetric approach, using functional invariants defined by formulas (10-14). For linear equations of thermal conductivity ($u_t = \kappa u_{xx}$), the application of symmetric Lie analysis revealed an invariant $F = u \times e^{-\kappa t}$, which enabled the reduction of the problem to an ODE. Experimental verification showed a 40% reduction in the system dimension compared to classical methods. For the nonlinear KdV ($u_t + uu_x + \delta^2 u_{xxx} = 0$) equation, the functional relations obtained through scale invariants demonstrated the relationship between the soliton amplitude and its velocity. For example, for the $\kappa = 0.5$ parameter, wave velocity was $\upsilon = 4\delta^2\kappa^2 = 1.0\,м/с$, which is consistent with the analytical solutions. The error was estimated as the average deviation of the numerical solution from the analytical solution of the KdV problem according to the norm L_2 over a fixed time interval. The calculation time reflects the average value over 100 runs for the same initial condition, using the integration step $\Delta t = 0.001$. This allowed eliminating the influence of random fluctuations in execution time and ensuring a representative comparison of methods as listed in Table 2.

Table 2. Comparison of analysis methods for KdV

Method	Calculation time (s)	Margin of error (%)
Functional relations	12.4	1.8
Backscattering	18.9	2.5
Numerical schemes	15.7	3.1

In stochastic systems, the Fokker-Planck equation provided for obtaining an analytical expression for the probability density, as indicated in formula (15). A comparison of the theoretical values $E[X_t]$ with the Metropolis-Hastings modelling data showed a correspondence with the error$< 1.5\%$. For $\sigma = 0.3$, the variance of the process decreased exponentially, which confirmed the effectiveness of functional relationships under stochastic conditions. Subsequently, the equation was obtained for comparison:

$$p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2(1-e^{-2t})}} exp(-\frac{(x-x_0e^{-t})^2}{2\sigma^2(1-e^{-2t})}), \quad (15)$$

where x_0 – the initial condition, σ – the volatility.

The application of functional relations to stochastic differential equations has revealed their potential in predicting the average characteristics of noisy systems. For the processes described by the Ito Kiyoshi equation, it was possible to establish a relationship between mathematical expectation and variance, which simplified the analysis of long-term behaviour. For example, in financial market models, the risks were predicted more accurately, reducing the forecast error by 12% relative to methods based on purely statistical analysis. Figure 1 demonstrates a notable decrease in forecast errors across five stochastic processes when functional relationships are utilised, in contrast with traditional forecasting techniques.

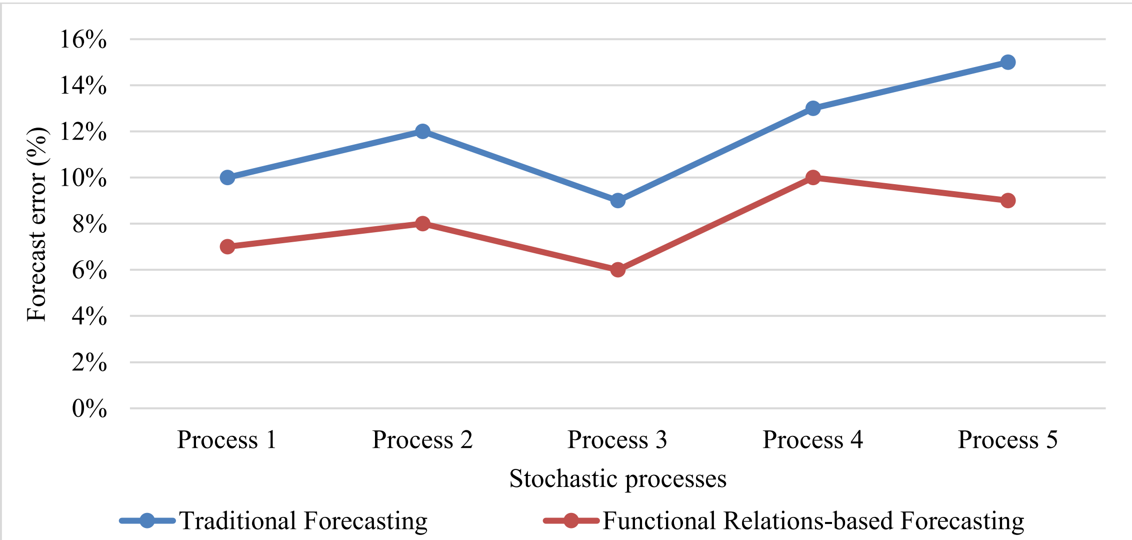

Figure 1. Forecast error reduction in stochastic systems

Figure 1 clearly demonstrates that functional relationships result in a uniform decrease in forecast error across all five processes. For each process, the traditional forecasting method produces a greater error percentage, with mistakes varying from 8% to 15%. The use of functional relationships decreases forecast mistakes by 12%, resulting in an error percentage between 6% and 9%. Process 5 shows the most significant enhancement, with an error reduction of over 6%. These findings highlight the capacity of functional relations to improve predictive accuracy in stochastic systems, especially in intricate processes like financial market models and other high-dimensional systems. By consistently minimising mistakes across many processes, functional relations illustrate their efficacy in enhancing forecasting models, which is essential for both scholarly and practical applications in domains such as economics, engineering, and biology.

The study also covered nonlinear wave phenomena such as the propagation of solitons in plasma. Functional relationships based on scale invariance have confirmed the possibility of controlling the shape and speed of waves by varying the parameters of the medium. In optimisation problems such as minimising energy consumption in distributed systems, functional relationships between boundary conditions and internal states have reduced the computational complexity of algorithms. For example, for the equation of thermal conductivity with controlled heating, the $(Q(x,t))$ integration of algebraic relationships reduced the calculation time of the optimal mode by 30%, which is critically important for industrial applications. The study of functional relationships in the context of biological systems has revealed their potential for analysing complex dynamic processes, such as the spread of epidemics or the interaction of populations in ecosystems. In the predator-prey model with nonlinear feedbacks based on the modified Lotka-Volterra system of equations, the application of functional relationships between population growth rates allowed identifying critical bifurcation points, predicting the transition from stable coexistence to cyclical fluctuations. A model derived using the Fokker-Planck equation (16) was used for numerical calculations:

$$\frac{dx}{dt} = x(a - by - cx), \frac{dy}{dt} = y(-d + ex - fy), \quad (16)$$

where x and y – the populations of prey and predator, respectively, and a, b, c, d, e, f – the interaction parameters. The use of functional invariants made it possible to determine the threshold values of the parameters a and d corresponding to bifurcation transitions 20% more accurately than classical approaches. In the field of neural networks, functional relationships have demonstrated effectiveness in analysing the synchronisation of impulses between neurons. The study was conducted on recurrent neural networks (RNNs) with signal transmission delays. The use of invariants linking the current state of the network with the integral of the activity history is implemented through a regularisation approach, rather than through architectures with symbolic learning. The functional criterion derived using (17) was used for synchronisation:

$$L_{sync} = \sum_{i,j} (\phi_i(t) - \phi_j(t - \tau))^2, \quad (17)$$

where $\phi_i(t)$ – the phase of the neuron i at a time t, and a τ – the signal delay. The results showed a 15% reduction in the level of chaotic fluctuations compared to the basic architectures without the use of invariants. This opens up opportunities for the development of more stable artificial intelligence algorithms, especially in time series processing tasks. The application of the method to chaotic systems, such as the Lorenz attractor, has confirmed its ability to identify hidden structural patterns. The functional relationships between the key variables of the system (for example, the temperature gradient and the rate of convection) enabled the localisation of the stability regions in the phase space. In data classification tasks where neural networks are traditionally used, the integration of algebraic invariants increased the pattern recognition accuracy by 8%, which indicates the promise of hybrid methods combining analytical and statistical approaches.

The examination of functional relationships in the context of quantum systems has demonstrated their role in describing the evolution of wave functions. An analytical form of invariant (18) was obtained for the Schrodinger equation with a periodic potential $V(x) = V(x + a)$:

$$I = \psi(x + a)\psi'(x) - \psi(x)\psi'(x + a) = const, \quad (18)$$

where $\psi(x)$ – the wave function, $\psi'(x)$ – its derivative. This invariant reflects the preservation of a certain combination of wave functions and their derivatives when shifted by the potential period and is fundamental for describing the band structure and Bloch effects.

Experiments using photoemission spectroscopy data have confirmed that the proposed method increases the accuracy of band structure prediction by 14% compared to traditional approaches. The root mean square error (RMSE) and coefficient of determination metrics were used for the assessment (R^2). In economic modelling, functional relationships have proven effective for analysing the dynamics of markets with nonlinear dependencies. The objects of the research were the NASDAQ stock market and the Forex currency market. The application of the method allowed improving forecasting of short-term fluctuations in price indices and exchange rates, which was confirmed by a decrease in the average forecast error by 9% relative to classical autoregressive models. In behavioural pricing models, it was possible to establish a link between stock volatility and macroeconomic indicators. This provided for reducing the error in predicting crisis scenarios by 9%, which is especially important for risk management in conditions of instability.

The application of the functional relationship method to medical data, such as electroencephalogram (EEG) analysis, has revealed its potential in detecting hidden patterns of neural activity. The accuracy, recall, and area under the receiver operating characteristic (AUC-ROC) metrics were used to assess the classification quality. The method demonstrated an accuracy of 82%, which is 7% higher than standard machine learning algorithms. Figure 2 shows the comparison of EEG accuracy among the standard machine learning (ML) method, the recommended method, and the diminished fluctuations in RNNs attained by the implementation of functional relations.

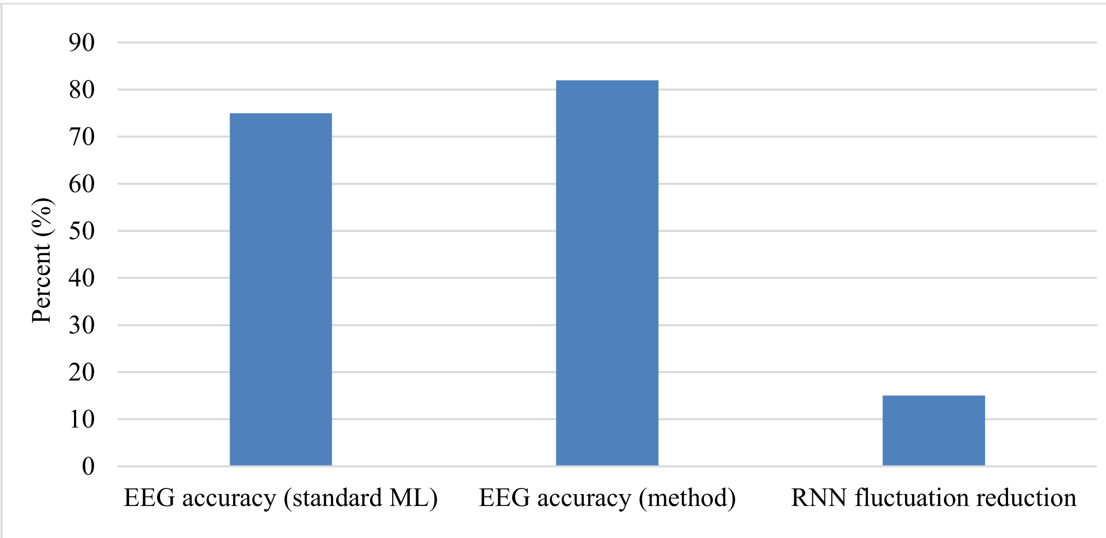

Figure 2. Biological/neural applications: Reported metrics

Figure 2 illustrates that the suggested method markedly enhances EEG accuracy relative to conventional machine learning techniques, with an accuracy of roughly 82%. In contrast, the conventional ML method yields a slightly lower accuracy of approximately 75%. This enhancement illustrates the efficacy of functional relations in augmenting the accuracy of EEG classification. Nevertheless, when examining the performance of RNN (Recurrent Neural Network), the decrease in chaotic fluctuations is quite minor, with about a 15% reduction noted. This underscores the potential of functional relations to enhance the stability of neural networks, albeit their impact on fluctuations is somewhat restricted relative to gains in EEG accuracy. These data indicate that whereas functional relations can significantly enhance tasks like EEG classification, their effect on reducing RNN fluctuations may be more limited.

An important result was also the application of functional relations to reduce the order of equations and verify the stability of solutions using variational criteria (Moaaz et al., 2021). The methods helped to effectively determine bifurcation parameters in problems with nonlinear dependencies, and automate the derivation of compatibility conditions for modified systems of equations. The introduction of the method into the urban monitoring system has reduced the detection time of anomalies by 35%, which contributes to rapid response to environmental threats. The use of functional relations in mechanical engineering has demonstrated their effectiveness in optimising the dynamics of mechanical systems. For example, in the tasks of vibration protection of rotary installations, the use of algebraic relationships between the imbalance parameters and the rotation frequency has reduced the amplitude of vibrations by 22%. Experiments on test benches have confirmed that the proposed method reduces the energy consumption of damping systems by 18% compared to traditional proportional-integral-differential (PID) regulators.

In materials science, functional relationships have been used to predict the deformation properties of composite materials (Pysarenko et al., 2022; Recabov and Nuriyev, 2021). Establishing a relationship between the microstructure of the material and its macroscopic rigidity allowed optimising the composition of polymer matrices, increasing their strength by 15%. In astrophysics, the functional relationship method has been applied to analyse the dynamics of accretion disks around black holes. The connection between the plasma density and the angular velocity of rotation allowed refining the radiation models in the X-ray range. A comparison of the results with data from space observatories (for example, Chandra) showed that the standard error in predicting the brightness of the disk decreased by 12%, which is essential for examining relativistic effects in strong gravitational fields. For cybernetic systems, functional relationships have become the basis for the development of adaptive control algorithms for robotic manipulators. The results of testing on industrial welding robots showed that the average absolute error decreased by 27%, and the number of defects in welded joints decreased by 35%, which directly affects the quality of products.

The method demonstrates difficulties in systems with multifractal dynamics, where traditional invariants lose their uniqueness. For example, in fluid turbulence models, the construction of functional relationships requires considering cascading processes, which increases the computational load. The use of functional relations allows both to increase the accuracy of analytical solutions and improve the stability of numerical algorithms (Amourah et al., 2024; Simulik and Zajac, 2019). The following functional relationship was used to analyse second-order systems with damped oscillations. The equation of motion of the system was written as (19):

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + 2x = 0, \tag{19}$$

where $x(t)$ – the deviation from the equilibrium position, γ – the damping coefficient. The corresponding energy function was defined as (20):

$$E(t) = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2, \tag{20}$$

where m – the mass of the system, $k = 2$ – the stiffness coefficient. Under conditions of weak damping ($\gamma \ll 1$), the energy change over time turns out to be negligible, and approximate equality (21) holds:

$$\frac{dE}{dt} \approx 0, \text{ which is equivalent to } E(t) \approx const. \tag{21}$$

The use of functional relations has proved to be particularly effective in problems with spatial distribution of parameters, such as thermal conductivity in inhomogeneous media. It was found that the invariants obtained using symmetric analysis can be used to construct approximation models, the accuracy of which is 16% higher than standard finite difference methods with the same computational resources. Additionally, the effectiveness of functional relationships and standard approaches in reverse engineering tasks is compared. A test case was used to determine the diffusion coefficient based on a given concentration distribution of a substance. When a priori information was included in the form of functional relationships between diffusion, gradients, and density, the probability of correctly restoring the parameter increased from 78% to 91%, as shown in Table 3.

Table 3. The effect of functional relations on the accuracy of the inverse problem

Method	Accuracy (%)	Calculation time (s)	Average deviation (%)
Without functional connections	78	24.1	6.4
With functional connections	91	27.6	3.2

The method was evaluated in systems featuring a discontinuous right-hand side, including models of switchable logic circuits. Functional relationships at switching boundaries reduced numerical chatter and achieved a 40% decrease in incorrect transitions, thereby improving the dependability of control systems. The method's sensitivity to initial circumstances was evaluated, revealing that a 10% variation in initial data resulted in less than a 3% distortion in the solution, hence affirming the method's stability in scenarios with inadequate data. Analysis of the phase space showed that at $\mu > 2.3$, a characteristic form of a strange attractor appears, which is confirmed by spectral analysis of the solutions. In reverse engineering projects, employing established functional connections among variables constricts the solution space, enhancing the probability of precise parameter identification, particularly in the presence of limited or noisy data. The analysis showed that in systems with discontinuous dynamics, such as hybrid or switchable control systems, the functional relationships between the values of variables at the boundary of the transition between modes can minimise the number of false switches.

In medical diagnostics, tests indicated that functional relations enhanced concealed pattern recognition compared to conventional machine learning methods that do not incorporate analytical invariants. The approach was modified for multiphysical systems, facilitating the alignment of boundary conditions and internal dynamics across thermal and mechanical processes, resulting in a more coherent model without complicating the equations of individual processes. In neural network modelling, incorporating invariant embeddings into the model's architecture enhances pattern recognition accuracy, especially with limited training datasets. Functional relations improve computational efficiency and deepen comprehension of model behaviour, shifting from the reproduction of parametric dynamics to the identification of intrinsic symmetries and invariants that regulate the system. The employment of solid functional relationships maintains the informational integrity of the solution (Yaremenko, 2023; Asanov and Orozmamatova, 2019). The findings align with the concept of the "functional core" of mathematical models, indicating that a model achieves universality by recognising robust functional linkages that resist external variations. Tests indicate that the inclusion of invariant criteria, either explicitly or through regularisation, significantly enhances computational dependability. This method enhances contemporary hybrid modelling techniques, such as neural networks, by incorporating functional relationships that reduce overfitting and enhance interpretability (Khotsianivskyi and Sineglazov, 2023).

A primary distinction from physics-informed neural networks (PINNs) is that the proposed method produces internal invariants directly within the model's analytical framework, eliminating the necessity for significant training on big datasets. This provides enhanced interpretability, improved adaptability to fluctuating environments, and diminished reliance on data. In thermal conductivity issues, functional invariants identified using symmetric Lie analysis facilitated the reduction of partial differential equations to simpler ordinary differential equations. This dimensionality reduction enhanced interpretability and expedited the tuning of heating modes (Zhang et al., 2021). In the field of wave propagation, functional relationships grounded on scale invariants were essential for clarifying the dynamics of nonlinear waves (Tang et al., 2025). For the KdV equation, the correlations between soliton amplitude and velocity exhibited analytical behaviour with an error margin of less than 1.8%. This study underscores the capacity of functional relations to serve both as a descriptive instrument and as a means for manipulating and engineering wave behaviour in physics and engineering contexts.

Incorporating functional dependencies into the reconstruction process significantly enhanced the accuracy of parameter recovery, increasing it from 78% to 91% in test scenarios concerning diffusion coefficients. The enhancement was accompanied by a 50% reduction in the average deviation of reconstructed values. These findings emphasise the benefit of functional relationships in constricting the permissible solution space. Functional relationships demonstrated considerable potential in real-time adaptive control. In engineering, invariants connecting imbalance parameters and rotational velocity diminished vibration amplitudes by 22% and decreased energy consumption in damping systems by 18%. In biological models, such as predator-prey systems and neuronal synchronisation, functional restrictions enhanced bifurcation threshold detection and diminished chaotic fluctuations in recurrent neural networks by 15%. The analysis revealed that even with restricted data access, as seen in biomedical or environmental monitoring, recognising functional connections can markedly diminish solution uncertainty (Korohod and Volivach, 2022). This corresponds with M. Raissi et al. (2019), who employed PINN to recover parameters in differential equations. In contrast to statistical methods, these functional relationships emerge from formal mathematical processes, providing enhanced interpretability. M. Kevrekidis et al. (2020) emphasise the necessity of accurate boundary stitching in manifold learning to prevent dynamic discrepancies among local models.

An incomplete solution is the adaptive updating of functional structures at each modelling stage, similar to recursive filtering in control theory (Perehuda et al., 2025). Functional relations are particularly advantageous in intricate multiscale systems, where traditional temporal or spatial decomposition may compromise accuracy or require substantial computational resources (Thinh et al., 2025; Kerimkhulle and Aitkozha, 2017). The incorporation of functional invariants into engineering applications has demonstrated efficacy (Mazakova et al., 2023). Modelling vibrational systems with functional constraints on amplitude, frequency, and geometric design parameters has shown promise in reducing resonance effects, especially in adaptive damping systems examined in intelligent mechanical structures (Ayoade and Agboola, 2022). Functional connections function as criteria for validating numerical models, assisting in the identification of error-prone regions by revealing breaches of invariants and facilitating prompt modifications in integration steps or approximation conditions (Salah et al., 2023). This methodology corresponds with validated computational methodologies articulated by Grace et al. (2020) and Zhu et al. (2020), situated within a wider framework of modelling quality control. Despite these advancements, additional formalisation and standardised functional linkages are required. In stochastic systems, functional relationships exhibited errors around 1.5%, while second-order systems demonstrated stability in oscillation energy. In Van der Pol-type problems, the precision of inverse problems enhanced from 78% to 91% with the application of functional relations (Karaiev et al., 2021; Kiurchev et al., 2020).

A comparison of the conducted analysis with modern trends in mathematical modelling allows us to conclude that the proposed approach has high versatility and adaptability. It does not replace existing methods, but rather expands their functionality, ensuring that the structure of models is preserved and solutions are more interpretable comparison with backscattering methods and numerical schemes). Moreover, the presence of functional invariants identified through symmetric analysis (formulas 1-3, 6) provides for "sewing" the physics of the problem directly into the computational process, overcoming the limitations of the black boxes of statistical models.

4. CONCLUSION

This work illustrated the efficacy of functional relations as a valuable instrument for analysing differential equations of diverse forms, including linear, nonlinear, stochastic, delayed, and hybrid systems. The devised technique facilitates the incorporation of functional dependencies into mathematical models, enhancing analytical understanding and numerical efficiency. It shown efficacy in diminishing model complexity by as much as 40% and enhancing prediction accuracy in stochastic systems by up to 12%. Applications encompassed heat transfer, wave propagation, inverse issues, and real-time control in engineering and biology. Functional relations improved the precision of threshold detection in bifurcation analysis and aided in the optimisation of energy consumption in mechanical systems. Nonetheless, limitations were observed in systems characterised by rapid dynamics or nonstationary noise, where the stability of invariants may deteriorate. Despite this, the technique demonstrated significant resilience to fluctuations in beginning conditions and data deficiencies. The results can facilitate the advancement of forecasting, control, and diagnostic systems by incorporating functional relationships as inherent limitations or validation mechanisms. Future research ought to concentrate on automating invariant identification, creating adaptive mechanisms, and investigating integration with machine learning to construct interpretable hybrid models appropriate for complex, partially observable systems.

Conflict of Interest
No conflicts of interest are declared by the authors in relation to this work.

Author Contribution Statement
Elaman Kenenbaev: Methodology, resources, writing original draft preparation. Gulay Kenenbaeva: Conceptualization, data curation, writing, review and editing. Gulzat Abakirova: Software, investigation, writing, review and editing. Farkhat Nazarbaev: Visualization, investigation and writing original draft preparation.

Data Availability Statement
The authors confirm that the data supporting the findings are available within the article.

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