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Short Communication

A Class of Vertex Decomposable Flag Simplicial Complexes

Seyed Mohammad Ajdani[1] and Francisco Bulnes[2,3,4]*

[1]Department of Mathematics, Za. C., Islamic Azad University, Zanjan, Iran
[2]Department of Mathematics and Engineering, TESCHA, Chalco,
State of Mexico, 56641, Mexico
[3]INAMEI, Chalco, State of Mexico, 56600, Mexico
[4]Postgraduate Department, Selinus University of Science and Literature,
Bologna, 40121, Italy
*Corresponding author: francisco.bulnes@tesch.edu.mx

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ABSTRACT

In this paper, we investigate the vertex decomposability of clique complexes associated with simple graphs and establishes a structural characterization based on forbidden induced subgraphs. We prove that the clique complex $CL(G)$ is vertex decomposable if and only if the underlying graph G contains no induced subgraphs isomorphic to $2K_2$, C_4, or C_5. The proof proceeds by demonstrating that such graphs and their complements are chordal, implying that their independence complexes are vertex decomposable. Since the independence complex of the complement graph coincides with the clique complex of G, the desired result follows. Furthermore, important algebraic consequences are derived: every vertex decomposable complex is shellable and hence partitionable, implying the validity of Stanley's conjecture for the corresponding face ring $K[CL(G)]$. Thus, this work introduces a new class of vertex decomposable flag simplicial complexes arising from graphs free of $2K_2$, C_4, and C_5. The results provide a significant combinatorial framework connecting chordal graph theory, simplicial complex decomposability, and algebraic properties of Stanley-Reisner rings.

Keywords: Vertex decomposable complex; Clique complex; Chordal graph; Flag simplicial complex; Stanley's conjecture; Partitionable complex

1. INTRODUCTION

The study of simplicial complexes arising from graphs has long served as a bridge between combinatorial topology and commutative algebra. Among the various structural properties of simplicial complexes, vertex decomposability plays a fundamental role due to its strong implications for shellability, partitionability, and algebraic behavior of the associated Stanley-Reisner rings. Understanding the conditions under which such combinatorial structures exhibit vertex decomposability provides deep insight into both the topological and algebraic nature of the underlying graphs (Björner, 1995; Conca and De Negri, 1999; Hachimori, 2008; He and Tuyl, 2010). In this context, clique complexes, constructed from the cliques of a graph, form an important subclass of simplicial complexes, especially as they are closely related to flag complexes. Previous work by Dochtermann and Engström (2009) showed that the independence complex of a chordal graph is vertex decomposable, establishing a crucial connection between chordal graph theory and simplicial topology. Building upon this framework, the present study focuses on characterizing graphs whose clique complexes possess this desirable property. We demonstrate that the clique complex $CL(G)$ is vertex decomposable precisely when the graph G contains no induced subgraphs isomorphic to $2K_2$, C_4, or C_5.

Let $R=K[x_1 ,\dots,x_n]$, where K is a field. Consider a simple graph G with vertex set $V(G)=\{1,\dots,n\}$ and edge set $E(G)$. The edge ideal of G is the quadratic squarefree monomial ideal defined as:

$$I(G)=(x_ix_j : ij \in E(G)) \subset R.$$

An independent vertex set of a graph G is a set of vertices in which no two vertices are adjacent. The independence complex of G, denoted by $\Delta(G)$, consists of all independent sets of G. The clique complex $CL(G)$ is the simplicial complex whose faces correspond to all complete subgraphs of G.

One of the central topics in combinatorial commutative algebra concerns Stanley's conjectures, which have been the subject of extensive research. For a N^n-graded ring R and a Z^n-graded R-module M, Stanley (1982) conjectured that:

$$\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$$

He further conjectured that every Cohen-Macaulay simplicial complex is partitionable. Herzog et al., (2008) later proved that the conjecture on partitionability is a special case of Stanley's first conjecture.

This paper is organized as follows. In Section 2, we recall essential definitions and preliminary results. In Section 3, we prove that the clique complex $CL(G)$ is vertex decomposable whenever G does not contain $2K_2$, C_4, or C_5 as an induced subgraph. As a consequence, we show that Stanley's conjecture holds for $K[CL(G)]$. Finally, we identify a class of vertex decomposable flag simplicial complexes, thereby extending the connection between graph theory, topology, and combinatorial algebra. This result not only extends known relationships between chordal graphs and decomposable complexes but also yields significant consequences in the validation of Stanley's conjecture and the structural classification of vertex decomposable flag simplicial complexes.

2. PRELIMINARIES

Definition 2.1 A simplicial complex Δ over a set of vertices $V = \{x_1, \dots, x_n\}$, is a collection of subsets of V, satisfying:

(a) $\{x_i\} \in \Delta$, for all i;

(b) if $F \in \Delta$, then every subset of F (including the empty set) is also in Δ.

An element of Δ is called a face, and complement of a face F is $F^c = V \setminus F$. The complement of the complex $\Delta = \langle F_1, \dots, F_r\rangle$ is $\Delta^c = \langle F_1^c, \dots, F_r^c\rangle$. The dimension of a face F is $\dim F = |F| - 1$, with $\dim \emptyset = -1$. Faces of dimension 0 and 1 are called vertices and edges, respectively. A non-face of Δ is a subset $F \subseteq V$ not belonging to Δ; minimal such subsets from the set $N(\Delta)$. Facets are maximal faces under inclusion, and $\dim \Delta$ is largest facet dimension. A complex is pure if all facets have the same dimension. If $F(\Delta) = \{F_1, \dots, F_q\}$ denotes the facet set, then $\Delta = \langle F_1, \dots, F_q\rangle$. A complex with one facet is a simplex. A subcomplex $\Gamma \subset \Delta$ satisfies $F(\Gamma) \subset F(\Delta)$. For a vertex $v \in V$, the deletion is:

$$\Delta \setminus v = \langle F \in \Delta : v \notin F\rangle.$$

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The link of a face F is

$$\text{Link\_}_{\Delta}(F) = \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\}.$$

For a vertex v,

$$\text{Link\_}_{\Delta}(v) = \{F \in \Delta : v \notin F, F \cup \{v\} \in \Delta\}.$$

Definition 2.2 Let Δ be a simplicial complex on the vertex set $\{x_1, . . . , x_n\}$. For any subset $F \subset \{x_1, . . . , x_n\}$, define:

$$x_F = \prod_{x_i \in F} x_i$$

The facet ideal of Δ, denoted by $I(\Delta)$, is the ideal of $S = K\{x_1, . . . , x_n\}$ generated by squarefree monomials $\{x_F : F \in F(\Delta)\}$. The non-face ideal or the Stanley-Reisner ideal of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{x_F : F \in N(\Delta)\}$, where $N(\Delta)$ is the set of minimal non-faces of Δ. The corresponding quotient ring

$$K[\Delta] := S / I\Delta$$

is called the Stanley-Reisner ring of Δ.

Definition 2.3 A simplicial complex Δ is said to be vertex decomposable if it satisfies one of the following recursive conditions:

(a) Δ is a simplex; or
(b) There exists a vertex v such that:

Both the deletion $\Delta \setminus v$ and the link $\text{link}_\Delta (v)$ are vertex decomposable; and No face of $\text{link}_\Delta (v)$ is a facet of $\Delta \setminus v$. A vertex v that satisfies condition (b) is called a shedding vertex of Δ.

Definition 2.4 A simplicial complex Δ is shellable if its facets can be ordered $F_1, . . . , F_s$ such that, for all $1 \le i < j \le s$, there exists some vertex $v \in F_j \setminus F_i$ and some $l \in \{1, . . . , j - 1\}$ with

$$F_j \setminus F_l = \{v\}.$$

Definition 2.5 A simplicial complex is a flag complex if all minimal non- faces have exactly two elements.

Definition 2.6 A graded S-module M is called sequentially Cohen-Macaulay (over K), if there exists a finite filtration of graded S-modules,

$$0 = M_0 \subset M_1 \subset \cdot \cdot \cdot \subset M_r = M$$

such that each $M_i/M_i{-}1$ is Cohen-Macaulay, and their Krull dimensions strictly increase:

$$\dim(M_1/ M_0) < \dim(M_2/ M_1) < \cdot \cdot \cdot < \dim(M_r/ M_{r-1})$$

A simplicial complex Δ is (sequentially) Cohen-Macaulay over K, if the Stanley–Reisner ring $K[\Delta] = S / I\Delta$ is (sequentially) Cohen-Macaulay. A simplicial complex Δ is disconnected, if the vertex set V can be partitioned into disjoint subsets $V_1 \cup V_2$ such that no face of Δ contains vertices from both V_1 and V_2. Otherwise, Δ is connected.

Definition 2.7 A graph G is chordal if every cycle of length strictly greater than three contains a chord, where a chord is an edge connecting two non-consecutive vertices in the cycle.

Definition 2.8 Let V_1 be a subset of the vertex set V of a graph G. The vertex-induced subgraph of G on V_1, denoted by $G[V_1]$, is the subgraph whose vertex set is V_1 and whose edge set consists of all edges of G with both endpoints in V_1.

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3. VERTEX DECOMPOSABILITY ON THE CLIQUE COMPLEXES

As the main result of this section, the clique complex $CL(G)$ associated with a simple graph G is vertex decomposable if G contains no induced subgraphs isomorphic to $2K_2$, C_4, or C_5. This result builds upon the work of Dochtermann and Engstrom (2009), who proved that for chordal graphs G, the independence complex $\Delta(G)$ is vertex decomposable. By demonstrating that the absence of these specific induced subgraphs ensures G and its complement G' are chordal, it follows that $\Delta(G') = CL(G)$ is vertex decomposable. Consequently, an important algebraic consequence is derived: Stanley's conjecture holds for the face ring $K[CL(G)]$. This theorem contributes significantly to the understanding of the interplay between graph-theoretic properties and topological-algebraic structures of the associated simplicial complexes.

3.1. Theorem - Vertex Decomposability of Clique Complexes

Let G be a simple graph. If G does not contain any induced subgraph isomorphic to $2K_2$, C_4, or C_5, then the clique complex $CL(G)$ is vertex decomposable.

Proof. Since any cycle C_n with $n > 5$ contains $2K_2$ as an induced subgraph, the absence of $2K_2$, C_4, and C_5 as induced subgraphs implies that G has no induced cycles longer than triangles (i.e., C_3). Hence, G is chordal. The same forbidden induced subgraph condition applies to the complement graph G', implying that G' is also chordal. This holds since the complements of $2K_2$, C_4, and C_5 are also of the same forbidden types, ensuring that the condition applies equally to G'.

By Dochtermann and Engström's (2009) result, the independence complexes $\Delta(G)$ and $\Delta(G')$ are vertex decomposable because G and G' are chordal. Since $\Delta(G') = CL(G)$, so $CL(G)$ inherits vertex decomposability. This theorem thus links forbidden induced subgraph conditions with strong topological and combinatorial properties of clique complexes associated with G.
Let R be a standard graded K-algebra over an infinite field K, i.e,

$$R = \bigoplus_{i\geq 0} R_i$$

with $R_0 = K$ and generated by R_1. The depth of R is the maximal length of a regular R-sequence consisting of linear forms. For $F \subseteq [n]$, define the sqaurefree monomial

$$x_F = \prod_{i\in F} x_i$$

and let $Z \subseteq \{x_1, \ldots, x_n\}$. The K-subspace

$$x_F\, K[Z] \subseteq S = K[x_1, \ldots, x_n]$$

is generated by monomials of the form $x_F u$, where u a monomial in the polynomial subring $K[Z] \subset S$. It is called a squarefree Stanley space if $\{x_i : i \in F\} \subseteq Z$. The dimension of this Stanley space is $|Z|$. Given a simplicial complex Δ on $\{x_1, \ldots, x_n\}$, a squarefree Stanley decomposition D of $K[\Delta]$ is a finite direct sum of squarefree Stanley spaces

$$K[\Delta] \cong \bigoplus_i u_i\, K[Z_i]$$

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The Stanley depth of a decomposition D, denoted sdepth(D), is the minimal dimension of the Stanley spaces in D. The Stanley depth of $K[\Delta]$ is defined as

$$\mathrm{sdepth}(K[\Delta]) = \max\{\mathrm{sdepth}(D)\} \mid D \text{ is a Stanley decomposition of } K[\Delta]\}.$$

Stanley (1982) conjectured that for any such ring,

$$\mathrm{depth}(K[\Delta]) \leq \mathrm{sdepth}(K[\Delta]).$$

Furthermore, Stanley conjectured that every Cohen–Macaulay simplicial complex is partitionable. A simplicial complex Δ with facets $G_1,\ldots,G_t$ is partitionable if it admits a partition

$$\Delta = \bigsqcup_{i=1}^{t} [F_i, G_i]$$

where each $[F_i, G_i]$ is the inerval of faces satisfying $F_i \subseteq H \subseteq G_i$.

Herzog et al., (2008) and Herzog and Hibi, 2011 proved that for a Cohen-Macaulay simplicial complex Δ, the inequality $\mathrm{depth}(K[\Delta]) \leq \mathrm{sdepth}(K[\Delta])$ holds if and only if Δ is partitionable. Since vertex decomposable complexes are shellable and each shellable complex is partitionable, the vertex decomposability of a simplicial complex implies Stanley's conjecture holds for the associated ring. As a consequence of the results proved here regarding vertex decomposability of $CL(G)$ for graphs avoiding induced $2K_2$, C_4, or C_5, it follows that Stanley's conjecture holds for $K[CL(G)]$.

3.2. Corollary

Let G be a simple graph. If G does not contain a $2K_2$, C_4, or C_5 as an induced subgraph, then the clique complex $CL(G)$ is partitionable, and Stanley's conjecture holds for $K[CL(G)]$.
Proof. Since each vertex decomposable simplicial complex is shellable and every shellable complex is partitionable, the vertex decomposability result established in Theorem 3.1 immediately implies the partitionability of $CL(G)$. Consequently, Stanley's conjecture follows for $K[CL(G)]$. Moreover, it is well-known that the clique complex of any graph is flag. Thus, the following corollary identifies a class of vertex decomposable flag simplicial complexes. Let G be a simple graph. If G does not contain $2K_2$, C_4, or C_5 as induced subgraphs, then the clique complex $CL(G)$ is a vertex decomposable flag simplicial complex. These corollaries tie together important topological and algebraic properties, illustrating that graphs avoiding these specific induced subgraphs give rise to highly structured and well-behaved clique complexes.

4. CONCLUSION

In conclusion, the results in this section establish that if a simple graph G does not contain $2K_2$, C_4, or C_5 as induced subgraphs, then its clique complex $CL(G)$ is vertex decomposable. This follows from showing that both G and its complement G' are chordal, leading to vertex decomposability of their respective independence complexes. Since $CL(G)$ corresponds to $\Delta(G')$, the property transfers to the clique complex. As every vertex decomposable complex is shellable and every shellable complex is partitionable, Stanley's conjecture holds for the associated face ring $K[CL(G)]$. Consequently, the clique complexes of such graphs constitute a distinguished class of vertex decomposable, partitionable flag simplicial complexes.

Conflict of Interest
The authors declare no conflicts of interest

Author Contribution Statement
Seyed Mohammad Ajdani and Francisco Bulnes: Conceptualization, methodology, software, visualization, and investigation. Seyed Mohammad Ajdani: Data curation, and writing original draft. Francisco Bulnes: Supervision, reviewing and editing.

Data Availability Statement
The authors confirm that the data supporting the findings of this study are available within the article.

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