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Results Relating to Hirota Method and Singularity Analysis on Some Nonlinear Waves Equations

Keputusan Berkaitan Kaedah Hirota dan Analisis Kesingularan bagi Beberapa Persamaan Gelombang Tidak Linear

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Abstract

This article investigates on the connection between singularity analysis and Hirota method i.e. a direct method to obtain the multi-soliton solutions of nonlinear waves equations. This includes equations with single bilinear form and coupled system of bilinear forms, together with the use of Hirota *D*-operator and various types of transformation. In general, finding the proper substitution to obtain the Hirota bilinear form is not an easy task. Singularity analysis is used to formulate this suitable transformation. This analysis is applied to Korteweg-de Vries (KdV), modified KdV and nonlinear Schröedinger (NLS) equations.

Keywords Hirota method, singularity analysis, *D*-operator

Abstrak

Artikel ini menyelidiki kaitan antara analisis kesingularan dan kaedah Hirota, iaitu satu kaedah langsung untuk memperoleh penyelesaian soliton berbilang bagi persamaan gelombang tidak linear. Persamaan-persamaan ini termasuklah yang mempunyai bentuk bilinear tunggal dan sistem bentuk bilinear berpasangan, dengan penggunaan pengoperasi *D* Hirota dan berbagai jenis transformasi. Secara am, untuk mendapatkan gantian sesuai bagi memperoleh bentuk bilinear Hirota adalah tugas yang bukan mudah. Analisis kesingularan digunakan untuk memformulasi transformasi yang sesuai itu. Analisis ini digunakan terhadap persamaan Korteweg-de Vries (KdV), persamaan KdV terubahsuai dan persamaan Schröedinger tidak linear (NLS).

Kata kunci Kaedah Hirota, analisis kesingularan, pengoperasi *D*

Introduction

The study of soliton theory is always a major source of mathematical and physical inspiration. For the past few decades, soliton theory has attracted considerable attention in diverse physical applications and the various mathematical methods of solution (e.g. Ablowitz & Clarkson, 1991, Infeld & Rowlands, 2001). In the 1970s, Hirota (1971) developed an ingenious method that is geared to finding multi-soliton solutions to nonlinear evolution equations directly. Although the method is less general than the Inverse Scattering Technique (IST) (Gardner *et al*., 1967, Bullough & Caudrey, 1980, Guo, 1995) since it does not solve initial-value problem, but it has the advantage of being applicable to a wider class of nonlinear equations in a unified way. In comparison to IST, Hirota method is rather heuristic, but it is more straightforward. We are of the opinion that if one is only interested in finding multi-soliton solutions, the best tool is Hirota method. In Hirota (2004), he discussed in details the process of finding the exact soliton solutions by using this algebraic method.

In this article, we investigate the connection between the singularity analysis and Hirota method (e.g. Gibbon *et al*., 1985). In general, finding the proper substitution to obtain the Hirota's bilinear form is not an easy task. Here we will discuss how the substitution can be obtained by means of singularity analysis, in particular by using the constructs in the Painlevé test (Chowdhury, 2000, Polyanin & Zaitsev, 2004). We may use the rational transformation

$$u = \frac{g}{f} \tag{1.1}$$

to transform the nonlinear differential equation into its bilinear equation. In fact, these choices of g and f can usually be done in such a way that movable singularities of u are zeroes of f. The movable singularities of the solution are the singularities of the solution (as a function of complex t) whose location depends on the initial conditions (Weiss *et al*., 1983). The system is said to possess the Painlevé property when all the movable singularities are simple poles.

If the equation has the generalized Painlevé property or at least the partial Painlevé property (Weiss, 1984, Tamizhmani *et al*., 2007), u has the Laurent expansion in the complex (x, t) space like in the Painlevé test. Then, the principle part of the expansion may be used for the substitution to obtain the Hirota's bilinear form. We apply this line of thinking to the Korteweg – de Vries (KdV), modified Korteweg – de Vries (mKdV) and nonlinear Schröedinger equations (NLS).

Theoretical Method and Results

In the following, we apply the above-mentioned method to three physically significant nonlinear waves equations, i.e. Korteweg – de Vries (KdV), modified Korteweg – de Vries (mKdV) and nonlinear Schröedinger (NLS) equations.

Korteweg – de Vries (KdV) Equation

Consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.2}$$

We expand locally in a generalized Laurent series of the form

$$u = f^{\gamma} \sum_{j=0}^{\infty} u_j f^j \tag{1.3}$$

with

$$f = f(x,t),$$
$$u_j = u_j(x,t)$$

Note that f and u are analytic functions of (x,t) in a neighbourhood of the manifold

$$M = \{(x,t) : f(x,t) = 0\},$$

and γ is an integer.

Substituting into (1.2), we have

$$\begin{aligned}&\gamma f^{\gamma-1} f_t u_0 + u_0 f^{\gamma} + 6 f^{\gamma} u_0 \left(\gamma f^{\gamma-1} f_x u_0 + u_{0x} f^{\gamma}\right) + \\ &f^{\gamma} u_{0xxx} + 3\gamma f^{\gamma-1} f_x u_{0xx} + 3\gamma(\gamma-1) f^{\gamma-2} f_x^2 u_{0x} + 3\gamma f^{\gamma-1} f_{xx} u_{0x} + \\ &3\gamma(\gamma-1) f^{\gamma-2} f_x f_{xx} u_0 + \gamma f^{\gamma-1} f_{xxx} u_0 + \gamma(\gamma-1)(\gamma-2) f^{\gamma-2} f_x^3 u_0 = 0\end{aligned} \tag{1.4}$$

Now, any value α which causes two or more dominant terms in equation (1.4) to balance and the rest of the terms to be ignored as $f \to 0$ is called a leading order, and the balancing terms are called the leading terms (e.g. Chowdhury, 2000).

Comparing the dominant terms of (1.4), i.e. $f^{\gamma-3}$ and $f^{2\gamma-1}$, we may obtain the leading order for (1.4) as

$$\gamma = -2$$

Now, inserting the expansion (1.3) into (1.2) we obtain the recursion relations as follows

$$f_x^2 u_j (j+1)(j-4)(j-6) = F\left(f_x, f_t; u_0, u_1, \ldots, u_{j-1}\right) \tag{1.5}$$

where F is some nonlinear functions of the derivatives of f and $u_i\left(j \le i \le j-1\right)$ with $j = 0, 1, 2, \ldots$. The "resonances" are at $j = -1, 4, 6$, where these occur when this recursion relation (1.5) becomes undefined.

The resonance at $j = 4$ introduces an arbitrary function u_4 and a compatibility condition on the functions (f, u_i, $i = 1, 2, 3$) which requires the right-hand side of the recursion relation to vanish identically as well as the resonance at $j = 6$. The resonance $j = -1$ corresponds to the arbitrariness of the singular manifold f. Except at $j = 4$ and $j = 6$, the rest can be found in terms of the previous u_j.

Now, since $\gamma = -2$, (1.3) may be written as

$$u = f^{\gamma} \sum_{j=0}^{\infty} u_j f^{j-2} \tag{1.6}$$

Then, we have for $0 \le j < \infty$,

$$u_t = u_{j,t} f^{j-2} + (j-2) f^{j-3} f_t u_j \tag{1.7}$$

$$u_x = u_{j,x} f^{j-2} + \left(j-2\right) f^{j-3} f_x u_j \tag{1.8}$$

$$\begin{aligned} u_{xxx} = {} & u_{j,xxx} f^{j-2} + \left(j-2\right) u_{j,xx} f^{j-3} f_x + 3\left(j-2\right)\left(j-3\right) u_{j,x} f^{j-4} f_x^2 + \\ & 3\left(j-2\right) u_{j,x} f^{j-3} f_{xx} + 3\left(j-2\right)\left(j-3\right) u_j f^{j-4} f_x f_{xx} + \\ & \left(j-2\right) u_{j,x} f^{j-3} f_{xxx} + \left(j-2\right)\left(j-3\right)\left(j-4\right) u_j f^{j-5} f_x^3 \end{aligned} \tag{1.9}$$

Substituting (1.7), (1.8) and (1.9) into (1.2), we have

$$\begin{aligned} & u_{j,t} f^{j-2} + \left(j-2\right) f^{j-3} f_t u_j + 6u_j f^{j-2} \left\{ u_{j,x} f^{j-2} + \left(j-2\right) f^{j-3} f_x u_j \right\} + \\ & u_{j,xxx} f^{j-2} + 3\left(j-2\right) u_{j,xx} f^{j-3} f_x + 3\left(j-2\right)\left(j-3\right) u_{j,x} f^{j-4} f_x^2 + \\ & 3\left(j-2\right) u_{j,x} f^{j-3} f_{xx} + 3\left(j-2\right)\left(j-3\right) u_j f^{j-4} f_x f_{xx} + \left(j-2\right) u_{j,x} f^{j-3} f_{xxx} + \\ & \left(j-2\right)\left(j-3\right)\left(j-4\right) u_j f^{j-5} f_x^3 = 0 \end{aligned} \tag{1.10}$$

Now, we expand (1.10) for j = 0, 1, 2, such that

$$\begin{aligned} & \left\{ u_{0,t} f^{-2} + u_{1,t} f^{-1} + u_{2,t} \right\} + \left\{ -2 f^{-3} + f_t u_0 - f^{-2} + f_t u_1 \right\} + \\ & \left\{ 6u_0 f^{-2} + 6u_1 f^{-1} + 6u_2 \right\} \left\{ u_{0,x} f^{-2} + u_{1,x} f^{-1} + u_{2,x} - 2 f^{-3} + f_x u_0 - f_x f^{-2} u_1 \right\} \\ & \left\{ + u_{0,xxx} f^{-2} + u_{1,xxx} f^{-1} + u_{2,xxx} \right\} + \left\{ -6 u_{0,xx} f^{-3} f_x - 3 u_{1,xx} f^{-2} f_x \right\} + \\ & \left\{ 18 u_{0,x} f^{-4} f_x^2 + 6 u_{1,x} f^{-3} f_x^2 \right\} + \left\{ -6 u_{0,x} f^{-3} f_{xx} - 3 u_{1,x} f^{-2} f_{xx} \right\} + \\ & \left\{ 18 u_{0,x} f^{-4} f_x f_{xx} + 6 u_1 f^{-3} f_x f_{xx} \right\} + \left\{ -2 u_{0,x} f^{-3} f_{xxx} - u_1 f^{-2} f_{xxx} \right\} + \\ & \left\{ -24 u_0 f^{-5} f_x^3 - 6 u_1 f^{-4} f_x^3 \right\} = 0 \end{aligned} \tag{1.11}$$

Collecting the terms with the same powers of f and equating the resulting coefficients to zero, we obtain

$$o\left(f^{-5}\right) : -12 u_0^2 f_x - 24 u_0 f_x^3 = 0 \tag{1.12}$$

$$o\left(f^{-4}\right) : 6 u_0 u_{0,x} - 18 u_0 u_1 f_x + 18 u_0 - 24 u_0 f_x f_{xx} - 6 u_1 f_x^3 = 0 \tag{1.13}$$

Thus, from (1.12), we obtain

$$u_{0,x} = -2 f_x^2 \tag{1.14}$$

Differentiating (1.14) with respect to x, we have

$$u_{0,x} = -4 f_x f_{xx} \tag{1.15}$$

Thus the ideas relating to singular analysis have been carried out successfully. Substituting (1.14) and (1.15) into (1.13), we then obtain

$$6\left(-2f_x^2\right)\left(-4f_xf_{xx}\right)-18\left(-2f_x^2\right)f_xu_1+18\left(-4f_xf_{xx}\right)f_x^2+$$
$$18\left(-2f_x^2\right)f_xf_{xx}-6f_x^3u_1=0 \tag{1.16}$$

Consequently,

$$-30f_x^3u_1=-60f_x^3f_{xx}$$

Thus,

$$u_1=2f_{xx} \tag{1.17}$$

If we proceed with the procedure successively, then we may obtain from (1.11) the following results:

$$j=0: u_0=-2f_x^2;$$
$$j=1: u_1=2f_{xx};$$
$$j=2: u_2=f_xf_t+6u_2f_x^2+4f_xf_{xxx}-3f_{xx}^2=0.$$

Substituting (1.14) and (1.15) into (1.3), we obtain the first two terms of u,

$$\begin{aligned} u &= u_0f^{-2}+u_1f^{-1} \\ &= u_0f^{-2}+u_1f^{-1} \\ &= \frac{2f_{xx}}{f}-\frac{2f_x^2}{f^2} \\ &= 2\left(\frac{ff_{xx}-f_x^2}{f^2}\right) \\ &= 2\left(log\, f\right)xx \end{aligned} \tag{1.18}$$

which is equivalent to the substitution in Hirota method, and with the "D" form we have

$$2\left(log\, f\right)xx=\frac{D_x^2f.f}{f^2} \tag{1.19}$$

depicting the connection between the singularity analysis and Hirota method, in finding the proper substitution (1.19).
Now, by using (1.18) as a proper choice of transformation, we will obtain

$$D_x\left(D_t+D_x^3\right)f.f=0,$$

as a simpler bilinear equation for (1.2).

Modified Korteweg – de Vries (mKdV) Equation

Following the procedure applied in the preceding section, we have for the mKdV equation of the form

$$u_t+24u^2u_x+u_{xxx}=0 \tag{1.20}$$

and similarly on applying the Laurent expansion (1.3) and the steps taken in section 2.1, this yields

$$\gamma = -1$$

$$u_0 = \pm \frac{i}{2} f_x$$

The solution may have two families of singularities, corresponding to the plus or minus sign in the dominant term. Thus, the substitution must set it free from both, and by using (1.6) and related steps as done in section 2.1, we have

$$\begin{aligned} u &= \frac{i}{2}\left(\frac{g_x}{g} - \frac{f_x}{f}\right) \\ &= \frac{i}{2}\left(\frac{fg_x - gf_x}{gf}\right) \\ &= \frac{i}{2}\left(\frac{f}{g}\right)\left(\frac{fg_x - gf_x}{f^2}\right) \\ &= \frac{i}{2}\left(log\frac{g}{f}\right)_x \end{aligned} \tag{1.21}$$

This is indeed equivalent to the classical substitution of Hirota method with the "*D*" form

$$\frac{i}{2}\left(log\frac{g}{f}\right)_x = \frac{i}{2}\left(\frac{D_x g.f}{gf}\right)$$

Since g is the complex conjugate of f, setting

$$f = F + iG$$

and

$$g = F - iG$$

we then obtain the substitution as

$$\begin{aligned} u &= \frac{i}{2}\left[log\frac{F-iG}{F+iG}\right]_x \\ &= \frac{i}{2}\frac{(F+iG)}{(F-iG)}\left\{\frac{(F+iG)(F_x - iG_x) - (F-iG)(F_x + iG_x)}{(F+iG)^2}\right\} \\ &= \left[\frac{FG_x - GF_x}{F^2+G^2}\right] \\ &= \left[tan^{-1}\frac{G}{F}\right]_x \end{aligned}$$

which is exactly as in Hirota method (Hirota, 1972 a, b).
Now, by using (1.21) as a proper choice of transformation, we will obtain

$$\left(D_t + D_x^3\right) g.f = 0 \tag{1.22a}$$

$$D_x^2 g.f + 2g^2 = 0 \tag{1.22b}$$

as a simpler bilinear equations for (1.20).

Nonlinear Schrödinger (NLS) Equation

The NLS equation is given by

$$i\psi_t + \psi_{xx} + |\psi|^2 \psi = 0 \tag{1.23}$$

In its expansion into a Laurent series, u and its complex conjugate v are no more conjugated and separated when x and t leave the real values. Therefore, the Laurent expansions for these functions are

$$\psi = f^{\gamma} \sum_{j=0}^{\infty} u_j f^j \tag{1.24a}$$

and

$$\phi = f^{\delta} \sum_{j=0}^{\infty} v_j f^j \tag{1.24b}$$

Following the same steps in sections 2.1 and 2.2, and applying the compatibility requirement of the dominant terms, we have

$$\gamma = \delta = -1 \tag{1.25a}$$

and

$$\psi_0 \phi_0 = -2 f_x^2 \tag{1.25b}$$

with one of ψ_0, ϕ_0 are to be arbitrary.
Let $\psi_0 = g$ and $\phi_0 = g^*$ where the asterisk denotes the complex conjugate.
Substituting

$$\psi = \frac{g}{f}$$

into (1.23), then we have too many unknown functions to be determined. Hence, we may use the coefficient (1.25a, b) of the Laurent expansion to associate g with f.
Replacing $\psi_0 = g$ and $\phi_0 = g^*$ in (1.25b), we have

$$gg^* = -2 f_x^2 \tag{1.26a}$$

Note that (1.26a) is in lacking of the "D" form as in (1.18) and (1.21). One can renormalize gg^* with a term of a higher order in the expansion variable f, i.e. one can carry out completion of the lower order terms with higher order ones to achieve the desired form of the coefficients; i.e. by adding $2 f f_{xx}$. Then we have

$$gg^* = -2 f_x^2 + 2 f\, f_{xx} \tag{1.26b}$$

Since we have known

$$D_x^2 f.f = 2f\,f_{xx} - 2f_x^2$$

then in this way we obtain

$$gg^* = 2f_x^2 + 2f\,f_{xx} = D_x^2 f.f \tag{1.27}$$

Indeed the bilinear forms of the NLS equation is equivalent to

$$f\left(iD_t + D_x^2\right)g.f - g\left(D_x^2 f.f - gg^*\right) = 0 \tag{1.28}$$

the requirement for this bilinear form is both the components of (1.28) vanish.
Now, by using (1.26) as a proper choice of transformation, we will obtain

$$f\left(iD_t + D_x^2\right)g.f = 0 \tag{1.29a}$$

$$g\left(D_x^2 f.f - gg^*\right) = 0 \tag{1.29b}$$

as a simpler bilinear equations for (1.23).

Conclusion

From the above discussion, clearly the Hirota method is definitely a useful tool for obtaining multi-soliton solutions, whiles the Laurent expansion and the singularity analysis can be used to make the equation bilinear by determining its principal part and completing it to the "*D*" form. From the bilinear form obtained, the perturbation method can be applied in order to find the exact soliton solutions for the nonlinear waves equations. Note that some of the nonlinear waves equations can be transformed into a single bilinear form but not all. Some of them will end up with a coupled system of bilinear equations. For this kind of bilinear form, finding the exact soliton solutions is more complicated since the solutions have to satisfy both constraints. The Hirota method can be used to construct the multi-soliton solutions for two equations from each category. For the single bilinear form, we have shown for the KdV equation whereas for the coupled system, we chose mKdV equation. The NLS equation calls for another consideration and restraint. Since there are a few types of transformation, we have simply discussed how the transformation can be chosen by means of singularity analysis. Thus, we have applied the singularity analysis to KdV, mKdV and NLS equations. The results showed that one may obtain simpler bilinear form using a suitable transformation which can be determined by means of singularity analysis.

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References

Ablowitz, M., & Clarkson, P. (1991). *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge: Cambridge University Press.

Bullough, R. K. & Caudrey, P. J. (1980). The Soliton and Its History. In Bullough, R.K. and Caudrey, P. J. (ed.) *Topics in Current Physics 17: Solitons.* New York: Spring-Verlag.

Chowdhury, A.R. (2000) *Painleve Analysis and Its Applications*, Chapman & Hall, London, UK.

Gardner, C. S., Greene, J. M., Kruskal, M. & Miura, R. M. (1967). Method for Solving the Korteweg-de Vries Equation. *Physical Review Letter*. 19(19), 1095–1097.

Gibbon, J. D., Radmore, P., Tabor, M. & Wood, D. (1985). The Painlevé Property and Hirota's Method. *Stud. App. Math.* 72, 39–63.

Guo, B. (1995). Soliton Theory and Modern Physics. In Gu, C. (ed.) *Soliton Theory and Its Application*. (pp. 1-64). Berlin: Springer and Zhejiang Science and Technology Publishing Hse.

Hirota, R. (1971). Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons. *Phys. Rev. Lett.* 27, 1192–1194.

Hirota, R. (1972a). Exact Solution of the Modified Korteweg-de vries Equation for Multiple Collisions of Solitons. *J. Phys. Soc. Jap.* 33(5), 1456–1458.

Hirota, R. (1972b). Exact Solution of the Sine-Gordon Equation for Multiple Collisions of Solitons. *J. Phys. Soc. Jap.* 33(5), 1459–1463.

Hirota, R. (2004). *The Direct Method in Soliton Theory*. Cambridge: Cambridge University Press.

Infeld, E., & Rowlands, G. (2001) *Nonlinear waves, solitons and chaos*, Cambridge: Cambridge University Press.

Polyanin, A. D. & Zaitsev, V. F. (2004). Painlevé Test for Nonlinear Equations of Mathematical Physics. In *Handbook of Nonlinear Partial Differential Equations.* New York: Chapman & Hall

Tamizhmani, K.M., Grammaticos, B., & Ramani, A., (2007). Do All Integrable Evolution Equations Have the Painlev´e Property? *SIGMA* 3 (2007), p. 73.

Weiss, J. (1984). On Classes of Integrable Systems and the Painlevé Property. *J. Math. Phys.* 25(1), 13–15.

Weiss, J., Tabor, M., & Carnevale, G. (1983). The Painlevé Property for Partial Differential Equations. *J. Math. Phys*. 24, 522–526.