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The Central Subgroup of the Nonabelian Tensor Square of the Second Bieberbach Group with Dihedral Point Group

Subkumpulan Pusat bagi Tensor Kuasa Dua Tak Abelan untuk Kumpulan Bieberbach Kedua dengan Kumpulan Titik Dwihedron

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Abstract

The properties of a group can be explored by computing the homological functors of the group. One of the homological functors of a group is the nabla, which is the central subgroup of the nonabelian tensor square. In this study, the nabla of the second Bieberbach group of dimension five with dihedral point group of order eight is computed. The abelianization of the group is first determined in order to compute its nabla.

Keywords homological functors, nabla, abelianization, Bieberbach group

Abstrak

Ciri-ciri suatu kumpulan boleh diterokai dengan mengira fungtor homologi kumpulan tersebut. Salah satu fungtor homologi bagi suatu kumpulan ialah nabla, iaitu subkumpulan pusat bagi tensor kuasa dua tak abelan. Dalam kajian ini, nabla bagi kumpulan Bieberbach kedua berdimensi lima dengan kumpulan titik dwihedron berdarjah lapan telah dikira. Abelanisasi bagi kumpulan tersebut telah ditentukan terlebih dahulu untuk mengira nablanya.

Kata kunci fungtor homologi, nabla, abelanisasi, kumpulan Bieberbach

INTRODUCTION

A Bieberbach group is defined as a torsion free crystallographic group which is given by a short exact sequence $1 \to L \to G \to P \to 1$ such that $G/\varphi L \cong P$ Here, L is called a lattice group and P is a finite point group. Since Bieberbach groups are crystallographic groups, any findings regarding these groups will give benefit to the chemists and physicists who are interested in the field of crystallography and spectroscopy. A Bieberbach group with dihedral point group has become an interest in this research where the group is explored by computing its homological functors such as the central subgroup of the nonabelian tensor square, denoted as $\nabla(G)$. The nonabelian tensor square $G \otimes G$ is a group generated by the symbols $g \otimes h,$ for all $g, h \in G,$ subject to relations $gh \otimes k = (g^h \otimes k^h)(h \otimes k)$ and $g \otimes hk = (g \otimes k)(g^k \otimes h^k)$ for all $g, h, k \in G$ where $g^h = h^{-1}gh$ (Brown & Loday, 1987). The $\nabla(G)$ is a normal subgroup generated by the element $g \otimes g$, for all $g \in G$ (Ellis, 1998). In order to compute $\nabla(G),$ the abelianization of the group is first determined since it plays an important role in the computation of the homological functor. The abelianization of the group, denoted as G^{ab} is a factor group of G / G', where G' is a derived subgroup.

Some research related to the computation of $\nabla(G)$ and G^{ab} of some Bieberbach groups have been done since 2009 starting with Rohaidah (2009) where she determined $\nabla(G)$ and G^{ab} of the Bieberbach groups with cyclic point group of order two. Nor'ashiqin & Nor Haniza (2010) also constructed the G^{ab} of the Bieberbach group of dimension four with dihedral point group of order eight. The result was used to compute $\nabla(G)$ of the group. Wan Nor Farhana et al. (2014) did the same work as Nor'ashiqin (2011) but with the first Bieberbach group of dimension five with dihedral point group of order eight. Besides that, Hazzirah Izzati et al. (2014) determined G^{ab} for all Bieberbach groups with cyclic point group of order three. Recently, Tan et al. (2014) have determined G^{ab} for all Bieberbach groups of dimension four with symmetric point group of order six. In this paper, the computation of $\nabla(G)$ and G^{ab} for the second Bieberbach group of dimension five with dihedral point group of order eight, denoted as $B_2(5),$ is presented.

PRELIMINARIES

This section provides some basic and structural results that been used in the computation of $\nabla(G)$ and G^{ab}. Definition 1 and 2 give the definition of polycyclic presentation and consistent polycyclic presentation, respectively.

Definition 1 (Eick & Nickel, 2008)

Let F_n be a free group on generators $g_1, ..., g_n$ and *R* be a set of relations of group *G*. The relations of a polycyclic presentation F_n / R have the form:

$$g_i^{e_i} = g_{i+1}^{x_{i,i+i}} ... g_n^{x_{i,n}} \qquad \text{for } i \le I,$$

$$g_j^{-1} g_i g_j = g_{j+1}^{y_{i,j,j+1}} ... g_n^{y_{i,j,n}} \qquad \text{for } j \le i,$$

$$g_j g_i g_j^{-1} = g_{j+1}^{z_{i,j,j+1}} ... g_n^{z_{i,j,n}} \qquad \text{for } j \le i \text{ and } j \notin I$$

for some $I \subseteq \{1, ..., n\}, e_i \in \mathrm{N}$ have $i \in I$ and $x_{i,j}, y_{i,j,k} z_{i,j,k} \in \mathrm{Z}$ for all *i, j* and *k*.

Definition 2 (Eick & Nickel, 2008)

Let *G* be a group generated by $g_1, ..., g_n$. The consistency of the relation in *G* can be determined using the following consistency relations:

$$g_k(g_j g_i) = (g_k g_j) g_i \qquad \text{for } k > j > i,$$

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$$(g_i^{e_j})g_i = g_j^{e_j^{-1}}(g_jg_i) \qquad \text{for } j > i, j \in I,$$

$$g_j(g_i^{e_i}) = (g_jg_i)g_i^{e_i^{-1}} \qquad \text{for } j > i, i \in I,$$

$$(g_i^{e_i})g_i = g_i(g_i^{e_i}) \qquad \text{for } i \notin I$$

$$g_j = (g_jg_i^{-1})g_i \qquad \text{for } j > i, i \notin I.$$

Theorem 1 shows some structural results of the nonabelian tensor square of group, provided by Blyth et al. (2010).

Theorem 1 (Blyth et al., 2010)

Let G be any group. Then

i. The nonabelian tensor square of any group G satisfies:

$$(G \otimes G)/K \cong \nabla(G/G') \times (G \wedge G),$$

where K is the kernel of the epimorphism $\nabla(G) \to \nabla(G/G')$.

ii. If G/G' has no element of order 2 then,

$$\nabla(G) \cong \nabla(G/G') \text{ and } \Gamma(G) \cong \nabla(G/G').$$

iii. If H is any finitely generated abelian group with independent generating set $\{a_1, a_2, ..., a_n\}$ then $(H \otimes H) \cong \nabla(H) \times (H \wedge H)$, where the independent generators is $\nabla(H)$ are the image of $(a_i \otimes a_i)$ for $i = 1,...,n$ and of $(a_i \otimes a_j)(a_j \otimes a_i)$ for all $1 \le i < j \le n$ and the independent generators of $H \wedge H$ are the image of $(a_i \otimes a_j)$ for all $1 \le i < j \le n$.

The group $v(G)$ has been introduced by Rocco (1991) and its definition is given as follows:

Definition 3 (Rocco, 1991)

Let G be a group with presentation $\langle \mathcal{G} | \mathcal{R} \rangle$ and let G^φ be an isomorphic copy of G via the mapping $\varphi : G \to G^\varphi$ for all $g \in G$. The group $v(G)$ is defined to be

$$v(G) = \left\langle \mathcal{G}, \mathcal{G}^\varphi \,\middle|\, \mathcal{R}, \mathcal{R}^\varphi, {}^x[g, h^\varphi] = [{}^x g, ({}^x h)^\varphi] = {}^{x^\varphi}[g, h^\varphi] \right\rangle, \forall x, g, h \in G.$$

Theorem 2 shows that the commutator subgroup of $v(G)$ is isomorphic to the nonabelian tensor square of group G. Therefore, the computation of $\nabla(G)$ can be done by using the commutator subgroup.

Theorem 2 (Ellis & Leonard, 1995)

Let G be a group. The map $\sigma : G \otimes G \to \left[G, G^{\varphi}\right] \lhd \nu(G)$ defined by $\sigma(g \otimes h) = \left[g, h^{\varphi}\right]$ for all g, h in G is an isomorphism.

Theorem 3 (Magidin & Morse, 2010)

Let G be any group. Then the natural homomorphism $\mu : G \to G/G'$ induces the epimorphism

$$f : [G, G^{\varphi}] \to (G/G') \otimes (G/G')$$

with $\left[x, y^{\varphi}\right] \mapsto \mu(x) \otimes \mu(y)$ for all x and y in G.

Lemma 1 (Rohaidah, 2009)

Let G be a group. If $c \in G$ is a commutator of the form $[x, y]$ then $\left[a, c^{\varphi}\right] = \left[c, a^{\varphi}\right]^{-1}$ in $\nu(G)$ for $a, x, y \in G$.

Theorem 4 (Blyth et al., 2010)

Let G be any group whose abelianization is finitely generated by the independent set $x_i G', i = 1, \ldots, n$. Let K be the kernel of the epimorphism $\nabla(G) \to \nabla(G/G')$ and let $E(G)$ be the subgroup of $\nu(G)$ defined by $E(G) = \left\langle \left[x_i, x_j^{\varphi}\right] \middle| 1 \le i < j \le n \right\rangle \left[G/(G')^{\varphi}\right]$. Then

i. $\nabla(G)$ is generated by the elements of the set $\left\{\left[x_i, x_i^{\varphi}\right], \left[x_i, x_j^{\varphi}\right]\left[x_i, x_i^{\varphi}\right] \middle| 1 \le i < j \le s\right\}$;

ii. $\nabla(G) \cap E(G) = K$ and $\nabla(G) E(G) = \left[G, G^{\varphi}\right]$.

Theorem 5 (Zomorodian, 2005)

Let *A, B* and *C* be any abelian group. Consider the ordinary tensor product of two abelian groups. Then,

i. $C_0 \otimes A \cong A,$

ii. $C_0 \otimes C_0 \cong C_0,$

iii. $C_n \otimes C_m \cong C_{\gcd(n,m)}$ for $n, m \in \mathbb{Z},$ and

iv. $A \otimes (B \otimes C) = (A \otimes B) \times (A \otimes C)$.

Main Results

A consistent polycyclic presentation of the group $B_2(5)$, which is determined based on Definition 1 and 2, is given as follows:

$$B_2(5)=\left\langle a,b,c,l_1,l_2,l_3,l_4,l_5 \;\middle|\; \begin{array}{l} a^2=l_4^{-1}, b^2=l_2^{-1}, c^2=l_1^{-1}, \\ b^a=cl_4^{-1}, c^a=bl_4^{-1}, c^b=cl_1l_2^{-1}l_3^{-1}, \\ l_1^a=l_2, l_2^a=l_1, l_3^a=l_3^{-1}, l_4^a=l_4, l_5^a=l_5, \\ l_1^b=l_1^{-1}, l_2^b=l_2, l_3^b=l_3^{-1}, l_4^b=l_4^{-1}, l_5^b=l_5^{-1}, \\ l_1^c=l_1, l_2^c=l_2^{-1}, l_3^c=l_3^{-1}, l_4^c=l_4^{-1}, l_5^c=l_5^{-1}, \\ l_j^{l_1}=l_j, l_j^{l_1^{-1}}=l_j, \text{for } j>i, 1\le i,j\le 5. \end{array}\right\rangle$$

The determination of the abelianization of $B_2(5)$ as in Lemma 2 and the computation of the nonabelian tensor square of the abelianization of $B_2(5)$ as in Lemma 3 are presented. The computation of $\nabla(B_2(5))$ are then shown.

Lemma 2

The abelianization of $B_2(5)$ is generated by the cosets $aB_2(5)'$ and $cB_2(5)'$ of order 4 and $l_5B_2(5)'$ of order 2. In particular, we write $B_2(5)^{ab}=B_2(5)/B_2(5)'\cong C_4^2\times C_2$.

Proof: From the relation of $B_2(5)$,

$$[a,l_1]=a^{-1}l_1^{-1}al_1=(l_1^{a})^{-1}l_1=l_2^{-1}l_1,$$
$$[a,l_2]=a^{-1}l_2^{-1}al_2=(l_2^{a})^{-1}l_2=l_1^{-1}l_2,$$
$$[a,l_3]=a^{-1}l_3^{-1}al_3=(l_3^{a})^{-1}l_3=l_3^{2},$$
$$[a,l_4]=a^{-1}l_4^{-1}al_4=(l_4^{a})^{-1}l_4=l_4^{-1}l_4=e,$$
$$[a,l_5]=a^{-1}l_5^{-1}al_5=(l_5^{a})^{-1}l_5=l_5^{-1}l_5=e,$$
$$[b,l_1]=b^{-1}l_1^{-1}bl_1=(l_1^{b})^{-1}l_1=l_1^{2},$$
$$[b,l_2]=b^{-1}l_2^{-1}bl_2=(l_2^{b})^{-1}l_2=l_2^{-1}l_2=e,$$
$$[b,l_3]=b^{-1}l_3^{-1}bl_3=(l_3^{b})^{-1}l_3=l_3^{2},$$
$$[b,l_4]=b^{-1}l_4^{-1}bl_3=(l_4^{b})^{-1}l_4=l_4^{2},$$
$$[b,l_5]=b^{-1}l_5^{-1}bl_5=(l_5^{b})^{-1}l_5=l_5^{2},$$
$$[c,l_1]=c^{-1}l_1^{-1}cl_1=(l_1^{c})^{-1}l_1=l_1^{-1}l_1=e,$$

$$[c,l_2]=c^{-1}l_2^{-1}cl_2=(l_2^{c})^{-1}l_2=l_2^2,$$
$$[c,l_3]=c^{-1}l_3^{-1}cl_3=(l_3^{c})^{-1}l_3=l_3^2,$$
$$[c,l_4]=c^{-1}l_4^{-1}cl_4=(l_4^{c})^{-1}l_4=l_4^2,$$
$$[c,l_5]=c^{-1}l_5^{-1}cl_5=(l_5^{c})^{-1}l_5=l_5^2,$$
$$[a,b]=a^{-1}b^{-1}ab=(b^{a})^{-1}b=l_4c^{-1}b,$$
$$[a,c]=a^{-1}c^{-1}ac=(c^{a})^{-1}c=l_4b^{-1}c \text{ and}$$
$$[b,c]=b^{-1}c^{-1}bc=(c^{b})^{-1}c=l_3l_2l_1^{-1}.$$

Then, $B_2(5)'=\langle bc, l_2^{-1}l_1, l_1^{-1}l_2, l_3, l_1^2, l_2^2, l_3^2, l_4^2, l_5^2\rangle$.

The abelianization $B_2(5)/B_2(5)'$ of $B_2(5)$ is generated by the cosets $aB_2(5)', bB_2(5)', cB_2(5)', l_1B_2(5)', l_2B_2(5)', l_3B_2(5)', l_4B_2(5)'$ and $l_5B_2(5)'$. However, since by relation of $B_2(5)$, we have $a^2=l_4^{-1}, b^2=l_2^{-1}, c^2=l_1^{-1}$, hence we have $l_1B_2(5)'=c^{-2}B_2(5)', l_2B_2(5)'=b^{-2}B_2(5)'$ and $l_4B_2(5)'=a^{-2}B_2(5)'$. Moreover, since we have $c^a=bl_4^{-1}$, hence $b=aca$, so $bB_2(5)'=(aca)B_2(5)'$. Since we also have $c^b=cl_1l_2^{-1}l_3^{-1}$, hence $l_3=b^{-1}c^{-1}bc^{-1}b^2=a^{-1}c^{-1}a^{-1}c^{-1}acac^{-1}a^2c^2a^2$. Thus $l_3B_2(5)'=(a^{-1}c^{-1}a^{-1}c^{-1}acac^{-1}a^2c^2a^2)B_2(5)'$.

Since we have $a^2=l_4^{-1}$, then $a^4=l_4^{-2}$. Furthermore, since $l_4^2\in B_2(5)'$ hence $l_4^{-2}\in B_2(5)'$. Similarly, we have $c^2=l_1^{-1}$, then $c^4=l_1^{-2}$. Since $l_1^2\in B_2(5)'$ so we have $l_1^{-2}\in B_2(5)'$. It follows that both $aB_2(5)'$ and $cB_2(5)'$ have order 4. Besides that, $l_5^2\in B_2(5)'$. So, $|l_5B_2(5)'|=2.$ Then, with all of the above, we have

$$B_2(5)^{ab}=B_2(5)/B_2(5)'\cong\langle aB_2(5)'\rangle\times\langle cB_2(5)'\rangle\times\langle l_5B_2(5)'\rangle\cong C_4\times C_4\times C_2\cong C_4^2\times C_2.$$

Next, the nonabelian tensor square of the abelianization of $B_2(5)$, namely $B_2(5)^{ab}\otimes B_2(5)^{ab}$ is determined as in Lemma 3.

Lemma 3

The nonabelian tensor square of abelianization of $B_2(5)$ is given as

$$B_2(5)^{ab}\otimes B_2(5)^{ab}\cong C_4^4\times C_2^5.$$

Proof: Let $\overline{a}=aB_2(5)'$, $\overline{c}=cB_2(5)'$ and $\overline{l_5}=l_5B_2(5)'$.

Hence by Theorem 1(iii), $B_2(5)^{ab}\otimes B_2(5)^{ab}$ is generated by

$\overline{a}\otimes\overline{a}$, $\overline{c}\otimes\overline{c}$, $\overline{l_5}\otimes\overline{l_5}$, $(\overline{a}\otimes\overline{c})(\overline{c}\otimes\overline{a})$, $(\overline{a}\otimes\overline{l_5})(\overline{l_5}\otimes\overline{a})$, $(\overline{c}\otimes\overline{l_5})(\overline{l_5}\otimes\overline{c})$, $\overline{a}\otimes\overline{c}$, and $\overline{a}\otimes\overline{l_5}$, $\overline{c}\otimes\overline{l_5}$. Hence, by Theorem 4 and Lemma 2,

$$\begin{aligned} B_2(5)^{ab} \otimes B_2(5)^{ab} &\cong C_4 \times C_4 \times C_2 \times C_4 \times C_2 \times C_2 \times C_4 \times C_2 \times C_2 \\ &\cong C_4^4 \times C_2^5. \end{aligned}$$

Next, we want to show the computation of $\nabla(B_2(5))$.

Theorem 6

Let $B_2(5)$ be a Bieberbach group of dimension five with dihedral point group of order eight. Then,

$$\begin{aligned} \nabla(B_2(5)) &= \left\langle [a,a^{\varphi}],[c,c^{\varphi}],[l_5,l_5^{\varphi}],[a,c^{\varphi}][c,a^{\varphi}],[a,l_5^{\varphi}][l_5,a^{\varphi}],[c,l_5^{\varphi}][l_5,c^{\varphi}] \right\rangle \\ &\cong C_8^2 \times C_4^2 \times C_2^2. \end{aligned}$$

Proof : By Lemma 2, $B_2(5)^{ab}$ is generated by $aB_2(5)'$, $cB_2(5)'$ and $l_5B_2(5)'$. Then by Theorem 4(i),

$$\nabla(B_2(5)) = \left\langle [a,a^{\varphi}],[c,c^{\varphi}],[l_5,l_5^{\varphi}],[a,c^{\varphi}][c,a^{\varphi}],[a,l_5^{\varphi}][l_5,a^{\varphi}],[c,l_5^{\varphi}][l_5,c^{\varphi}] \right\rangle.$$

Next, we need to show the order of $[a,a^{\varphi}]$ and $[c,c^{\varphi}]$ are 8, $[l_5,l_5^{\varphi}]$ and $[a,c^{\varphi}][c,a^{\varphi}]$ are of order 4 and $[a,l_5^{\varphi}][l_5,a^{\varphi}]$ and $[c,l_5^{\varphi}][l_5,c^{\varphi}]$ are of order 2. By relation of $B_2(5)$,

$$\begin{aligned} [a,a^{\varphi}]^{16} &= [a^4,a^{4\varphi}] \\ &= [l_4^{-2},l_4^{-2\varphi}] \\ &= [l_4^{2},l_4^{2\varphi}] \\ &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} [c,c^{\varphi}]^{16} &= [c^4,c^{4\varphi}] \\ &= [l_1^{-2},l_1^{-2\varphi}] \\ &= [l_1^{2},l_1^{2\varphi}] \\ &= 1. \end{aligned}$$

These mean that the order of $[a,a^{\varphi}]$ and $[c,c^{\varphi}]$ divide 16. Hence, the order of $[a,a^{\varphi}]$ and $[c,c^{\varphi}]$ maybe 2, 4, 8 or 16. By the epimorphism f as stated in Theorem 3, $f([a,a^{\varphi}]) = \mu(a)\otimes\mu(a) = \overline{a}\otimes\overline{a}$ and $f([c,c^{\varphi}]) = \mu(c)\otimes\mu(c) = \overline{c}\otimes\overline{c}$ and both have order 4 by Lemma 3. Hence $[a,a^{\varphi}]$ and $[c,c^{\varphi}]$ are of order a multiple of 4. So they are not of order 2.

The order of $[a,a^{\varphi}]$ and $[c,c^{\varphi}]$ cannot be 4 since:

$$\begin{aligned} [a,a^{\varphi}]^{4} &= [a^2,a^{2\varphi}] \\ &= [l_4^{-1},l_4^{-\varphi}] \\ &= [l_4,l_4^{\varphi}] \\ &\neq 1 \end{aligned} \quad \text{and} \quad \begin{aligned} [c,c^{\varphi}]^{4} &= [c^2,c^{2\varphi}] \\ &= [l_1^{-1},l_1^{-\varphi}] \\ &= [l_1,l_1^{\varphi}] \\ &\neq 1. \end{aligned}$$

Suppose $[a,a^{\varphi}]$ and $[c,c^{\varphi}]$ have order 16. Then, there is no positive integer r smaller than 16 such that $[a,a^{\varphi}]^{r}=1$. Since $a^4=l_4^{-2}\in B_2(5)'$ and $c^4=l_1^{-2}\in B_2(5)'$, then by Lemma 1,

$$\begin{aligned}[a,(a^4)^{\varphi}]&=[a^4,a^{\varphi}]^{-1} \\ [a,(a^4)^{\varphi}][a^4,a^{\varphi}]&=1 \\ [a,a^{\varphi}]^4[a,a^{\varphi}]^4&=1 \\ [a,a^{\varphi}]^8&=1\end{aligned} \quad \text{and} \quad \begin{aligned}[c,(c^4)^{\varphi}]&=[c^4,c^{\varphi}]^{-1} \\ [c,(c^4)^{\varphi}][c^4,c^{\varphi}]&=1 \\ [c,c^{\varphi}]^4[c,c^{\varphi}]^4&=1 \\ [c,c^{\varphi}]^8&=1\end{aligned}$$

which contradicts the facts that $[a,a^{\varphi}]$ and $[c,c^{\varphi}]$ are of order 16. Hence, $[a,a^{\varphi}]$ and $[c,c^{\varphi}]$ are of order 8. Besides that,

$$\begin{aligned}([a,c^{\varphi}][c,a^{\varphi}])^4&=[a,c^{4\varphi}][c^4,a^{\varphi}] \\ &=[a,l_1^{-2\varphi}][l_1^{-2},a^{\varphi}] \\ &=([a,l_1^{2\varphi}][l_1^{2},a^{\varphi}])^{-1} \\ &=([a,[b,l_1]^{\varphi}][[b,l_1],a^{\varphi}])^{-1} \\ &=([[b,l_1],a^{\varphi}]^{-1}[[b,l_1],a^{\varphi}])^{-1} \\ &=1.\end{aligned}$$

Hence, the order of $[a,c^{\varphi}][c,a^{\varphi}]$ divides 4, that is the order of $[a,c^{\varphi}][c,a^{\varphi}]$ is either 2 or 4. But the order of $[a,c^{\varphi}][c,a^{\varphi}]$ cannot be 2 since by Theorem 3,

$$\begin{aligned}f([a,c^{\varphi}][c,a^{\varphi}])&=f([a,c^{\varphi}])f([c,a^{\varphi}]) \\ &=(\mu(a)\otimes\mu(c))(\mu(c)\otimes\mu(a)) \\ &=(\overline{a}\otimes\overline{c})(\overline{c}\otimes\overline{a}).\end{aligned}$$

By Lemma 9, $f([a,c^{\varphi}][c,a^{\varphi}])$ has order 4. Hence the order of $[a,c^{\varphi}][c,a^{\varphi}]$ is a multiple of 4. Since the order of $[a,c^{\varphi}][c,a^{\varphi}]$ divides 4 but not 2, hence the order of $[a,c^{\varphi}][c,a^{\varphi}]$ is exactly 4.

Moreover, the order of $[l_5,l_5^{\varphi}]$ divides 4 since $l_5^2\in B_2(5)'$. Hence $[l_5,l_5^{\varphi}]^4=[l_5^{\,2},l_5^{\,2\varphi}]=1$. It means that the order of $[l_5,l_5^{\varphi}]$ is either 2 or 4. By Theorem 3, $f([l_5,l_5^{\varphi}])=\mu(l_5)\otimes\mu(l_5)=\overline{l_5}\otimes\overline{l_5}$ and has order 2 by Lemma 9. So, the order of $[l_5,l_5^{\varphi}]$ is at least 2. If the order of $[l_5,l_5^{\varphi}]$ is 2, then $[l_5,l_5^{\varphi}]^2=[l_5^{\,2},l_5^{\varphi}]=1$. However, the order of $[l_5,l_5^{\varphi}]$ cannot be 2 since there are no relation in $\nu(B_2(5))$ that allow l_5^2 and l_5^{φ} to commutes. Therefore, the order of $[l_5,l_5^{\varphi}]$ is 4.

By homomorphism in Theorem 3 and Theorem 5,

$$\langle \mu(a) \otimes \mu(l_5) \rangle \cong C_2,$$
$$\langle \mu(l_5) \otimes \mu(a) \rangle \cong C_2,$$
$$\langle \mu(c) \otimes \mu(l_5) \rangle \cong C_2,$$
$$\langle \mu(l_5) \otimes \mu(c) \rangle \cong C_2.$$

So, the order of $[a, l_5^{\varphi}][l_5, a^{\varphi}]$ and $[c, l_5^{\varphi}][l_5, c^{\varphi}]$ are 2. Therefore,

$$\nabla(B_2(5)) \cong C_8 \times C_8 \times C_4 \times C_4 \times C_2 \times C_2 \cong C_8^2 \times C_4^2 \times C_2^2.$$

This concludes the proof.

CONCLUSION

In this paper, the abelianization and the central subgroup of the nonabelian tensor square of the second Bieberbach group of dimension five with dihedral point group of order eight has been successfully computed. Based on this results, other homological functors such as the nonabelian tensor square, *G*-trivial subgroup of the nonabelian tensor square, denoted as $J(G)$, the nonabelian exterior square and Schur multiplier can be determined.

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