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On Computing the Nonabelian Tensor Square of a Bieberbach Group with Dihedral Point Group of Order Eight

Pengiraan Kuasa Dua Tensor Tak Abelan bagi satu kumpulan Bieberbech dengan Kumpulan Titik Dwihedron Peringkat Lapan

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Abstract

The nonabelian tensor square is one of the important homological functors of a group**.** A Bieberbach group is a torsion free crystallographic group. In this paper, we compute the nonabelian tensor square of a Bieberbach group of dimension five with dihedral point group of order eight.

Keywords nonabelian tensor square, Bieberbach group

Abstract

Kuasa dua tensor tak abelan merupakan salah satu fungtor homologi yang penting bagi suatu kumpulan. Kumpulan Bieberbach ialah satu kumpulan kristalografi yang bebas kilasan. Dalam artikel ini kuasa dua tensor tak abelan untuk satu kumpulan Bieberbach berdimensi lima dengan kumpulan titik dwihedron berperingkat lapan telah dihitung.

Keywords kuasa dua tensor tak abelan, kumpulan Bieberbach

INTRODUCTION

The nonabelian tensor product $G \otimes H$ for arbitrary groups *G* and *H* has been introduced by Brown and Loday (1987). The nonabelian tensor product $G \otimes H$ of groups *G* and *H* is generated by the symbols $g \otimes h,$ for all $g \in G,\ h \in H,$ subject to relations

$$gg' \otimes h' = (g^{g'} \otimes h^{g})(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)(g^{h} \otimes h^{h'})$$

for all $g, g' \in G$ and $h, h' \in H,$ where $g^{g'} = g^{-1}g'g$. A special case of the nonabelian tensor product is the product of two similar groups, known as the nonabelian tensor square. The nonabelian tensor square of a group *G,* denoted by $G \otimes G$.

Bieberbach groups are torsion free crystallographic groups. These groups are extension of a finite point group *P* and a free abelian group *L* of finite rank. Throughout the years, the nonabelian tensor square has been investigated by many researchers. Brown et al. (1987) started the investigation of the nonabelian tensor square for a finite group by forming the finite presentation. While Blyth et al. (2004) studied the nonabelian tensor squares of the free 2-Engel groups. In 2009, Blyth et al. studied some structural results on the nonabelian tensor square of finite p-groups. The study of the nonabelian tensor squares of certain Bieberbach groups with cyclic point groups was started by Rohaidah (2009). This was followed by the computation of the nonabelian tensor square of the Bieberbach group of dimension four with dihedral point group of order eight (Nor'ashiqin & Nor Haniza, 2010). Wan Nor Farhana et al. (2014a) did the same work as Nor'ashiqin and Nor Haniza (2010) but the computation was on the first Bieberbach group of dimension five with dihedral point group of order eight, denoted by $B_1(5)$.

The main focus of this paper is to compute the nonabelian tensor square of the second Bieberbach group of dimension five with dihedral point group of order eight, denoted by $B_2(5)$. It means that, we need to find the generators of $B_2(5)$ and its presentation. So the main theorems that need to be proved in this paper are the following Theorem 1 and Theorem 2.

Theorem 1

Let $B_2(5)$ be the second Bieberbach group of dimension five with the dihedral point group of order eight. Then

$$B_2(5)\otimes B_2(5)=\left\langle \begin{matrix} a\otimes a, c\otimes c, l_5\otimes l_5, a\otimes b, a\otimes c, a\otimes l_1, a\otimes l_5, b\otimes l_1, b\otimes l_4, c\otimes l_5, \\ (a\otimes c)(c\otimes a), (a\otimes l_5)(l_5\otimes a), (c\otimes l_5)(l_5\otimes c) \end{matrix} \right\rangle.$$

Theorem 2

Let $B_2(5)$ be the second Bieberbach group of dimension five with the dihedral point group of order eight. Then the presentation of the nonabelian tensor square, $B_2(5)\otimes B_2(5)$ of the group $B_2(5)$ is given as follows:

$$B_2(5)\otimes B_2(5)=\left\langle g_1, g_2, \ldots, g_{13} \;\middle|\; \begin{matrix} g_1^8=g_2^8=g_3^4=g_4^4=g_5^2=g_6^2=[g_7,g_8]=[g_7,g_{10}]=[g_7,g_{12}]=[g_8,g_{10}] \\ =[g_8,g_{12}]=[g_9,g_{10}]=[g_9,g_{11}]=[g_9,g_{12}]=[g_9,g_{13}]=[g_{10},g_{11}]=[g_{10},g_{12}] \\ =[g_{10},g_{13}]=[g_{11},g_{12}]=[g_{11},g_{13}]=1, [g_7,g_9]=[g_8,g_9]=g_2^{-4}g_9^2, [g_7,g_{11}] \\ =[g_8,g_{11}]=g_2^4 g_{11}^2, [g_7,g_{13}]=[g_8,g_{13}]=g_{10}^{-4}, [g_{12},g_{13}]=g_{10}^{-8}, \text{ for } 1\le i\le 6, 1\le j\le 13 \end{matrix} \right\rangle, \quad (1)$$

where

$a\otimes a=g_1$, $c\otimes c=g_2$, $l_5\otimes l_5=g_3$, $(a\otimes c)(c\otimes a)=g_4$, $(a\otimes l_5)(l_5\otimes a)=g_5$, $(c\otimes l_5)(l_5\otimes c)=g_6$,
$a\otimes b=g_7, a\otimes c=g_8$, $a\otimes l_1=g_9$, $a\otimes l_5=g_{10}$, $b\otimes l_1=g_{11}$, $b\otimes l_4=g_{12}$, $c\otimes l_5=g_{13}$.

PRELIMINARY RESULTS

This section provides some basic definitions and structural results used in this paper.

The consistency polycyclic presentation of $B_2(5)$ (Wan Nor Farhana et al., 2014b) is given as follows:

$$B_2(5)=\left\langle a,b,c,l_1,l_2,l_3,l_4,l_5 \;\middle|\; \begin{array}{l} a^2=l_4^{-1}, b^2=l_2^{-1}, c^2=l_1^{-1}, \\ b^a=cl_4^{-1}, c^a=bl_4^{-1}, c^b=cl_1l_2^{-1}l_3^{-1}, \\ l_1^a=l_2, l_2^a=l_1, l_3^a=l_3^{-1}, l_4^a=l_4, l_5^a=l_5, \\ l_1^b=l_1^{-1}, l_2^b=l_2, l_3^b=l_3^{-1}, l_4^b=l_4^{-1}, l_5^b=l_5^{-1}, \\ l_1^c=l_1, l_2^c=l_2^{-1}, l_3^c=l_3^{-1}, l_4^c=l_4^{-1}, l_5^c=l_5^{-1}, \\ l_j^{l_i}=l_j, l_j^{l_i^{-1}}=l_j, \text{for } j>i, 1\le i,j\le 5. \end{array} \right\rangle. \quad (2)$$

The structural results of $\nu(G)$, which is a subgroup of $[G,G^{\varphi}]$, was studied by Ellis and Leonard (1995) and Rocco (1991) and has been extended by Blyth and Morse (2009). The definition of $\nu(G)$ is given as follows.

Definition 1 (Rocco, 1991)

Let *G* be a group with presentation $\langle G|R\rangle$ and let G^{φ} be an isomorphic copy of *G* via the mapping $\varphi: g\to g^{\varphi}$ for all $g\in G$. The group $\nu(G)$ is defined to be

$$\nu(G)=\left\langle G,G^{\varphi} \;\middle|\; R,R^{\varphi}, {}^{x}[g,h^{\varphi}]=[{}^{x}g,({}^{x}h)^{\varphi}]={}^{x\varphi}[g,h^{\varphi}], \forall x,g,h\in G\right\rangle.$$

The following theorem shows that $G\otimes G$ is isomorphic to $[G,G^{\varphi}]$ of the group *G*.

Theorem 3 (Ellis & Leonard, 1995)

Let *G* be a group. The map $\sigma: G\otimes G\to [G,G^{\varphi}]\triangleleft \nu(G)$ defined by $\sigma(g\otimes h)=[g,h^{\varphi}]$ for all *g, h* in *G* is an isomorphism.

The following commutator calculus and lemmas are used in the computation of the nonabelian tensor square of $B_2(5)$. For right conjugation, $y^x=x^{-1}yx$, $[x,y]=x^{-1}y^{-1}xy$ and $[x,y,z]=[[x,y],z]$, the following identities hold:

$$[xy,z]=[x,y]^{y}\cdot[y,z]=[x,z]\cdot[[x,z],y]\cdot[y,z]; \quad (3)$$

$$[x,yz]=[x,z]\cdot[x,y]^{z}=[x,z]\cdot[x,y]\cdot[[x,y],z]; \quad (4)$$

$$[x,y]=[x,y^{-1}]^{-y}=[x^{-1},y]^{-x}; \quad (5)$$

$$[x^{-1},y]=[x,y]^{-x^{-y}}=[x^{-1},[x,y]].[x,y]^{-1}; \quad (6)$$

$$[x,y^{-1}]=[x,y]^{-y^{-x}}=[y^{-1},[x,y]].[x,y]^{-1}. \quad (7)$$

Lemma 1 (Blyth et al., 2008; Blyth & Morse, 2009)

Let G be a group. The following relations hold in $\nu(G)$:

i. $[g_1,[g_2,g_3]^{\varphi}]=[g_2,g_3,g_1{}^{\varphi}]^{-1}$ for all g_1,g_2,g_3 in G;
ii. $[g,g^{\varphi}]$ is a central in $\nu(G)$ for all g in G;
iii. $[g_1,g_2{}^{\varphi}][g_2,g_1{}^{\varphi}]$ is a central in $\nu(G)$ for all g_1,g_2 in G;
iv. $[g,g^{\varphi}]=1$ for all g in G'.

Lemma 2 (Blyth et al., 2008; Blyth & Morse, 2009)

Let x and y be element of G such that $[x,y]=1$. Then in $\nu(G)$,

i. $[x^n,y^{\varphi}]=[x,y^{\varphi}]^n=[x,(y^{\varphi})^n]$ for all integer n;
ii. $[x^n,(y^m)^{\varphi}][y^m,(x^n)^{\varphi}]=([x,y^{\varphi}][y,(x^{\varphi})]^{nm}$;
iii. $[x,y^{\varphi}]$ is a central in $\nu(G)$;

Lemma 3 (Blyth et al., 2008; Blyth & Morse, 2009)

Let g_1,g_2,g_3 and g_4 be elements of a group G. Then in $\nu(G)$,

i. $[[g_1,g_2{}^{\varphi}],[g_2,g_1{}^{\varphi}]]=1$;
ii. $[[g_1,g_2][g_3,g_4]^{\varphi}]=[[g_1,g_2{}^{\varphi}],[g_3,g_4{}^{\varphi}]]$;
iii. $[g_1^n,g_2^{\varphi}].[g_2,(g_1^n)^{\varphi}]=[g_1,(g_2^n)^{\varphi}].[g_2^n,g_1{}^{\varphi}]=([g_1,g_2{}^{\varphi}][g_2,g_1{}^{\varphi}])^n$.

In order to determine the structure of $G\otimes G$, the subgroup of $\nu(G)$ need to be computed and the following proposition is used to compute the subgroup of $\nu(G)$.

Proposition 1 (Blyth & Morse, 2009)

Let G be a polycyclic group with a polycyclic generating sequence $g_1,\ldots,g_k$. Then $[G,G^{\varphi}]$, a subgroup of $\nu(G)$, is generated by $[G,G^{\varphi}]=\langle[g_i,g_i{}^{\varphi}],[g_i^{\varepsilon},(g_j{}^{\varphi})^{\delta}],[g_i^{\varepsilon},(g_j{}^{\varphi})^{\delta}][g_j^{\delta},(g_i{}^{\varphi})^{\varepsilon}]\rangle$ for $1\le i<j\le k$, where

$$\varepsilon=\begin{cases}1 & \text{if } |g_i|<\infty\\ \pm1 & \text{if } |g_i|=\infty\end{cases} \quad\text{and}\quad \delta=\begin{cases}1 & \text{if } |g_j|<\infty\\ \pm1 & \text{if } |g_j|=\infty\end{cases}.$$

COMPUTING THE NONABELIAN TENSOR SQUARE

Using the theory of the computation of $G\otimes G$ of polycyclic group by Blyth & Morse (2009), the independent generators and the presentation of $B_2(5)\otimes B_2(5)$ of $B_2(5)$ are computed. By Proposition 1, the subgroup $[B_2(5),B_2(5)^{\varphi}]$ which is isomorphic to $B_2(5)\otimes B_2(5)$ is generated by 148 generators which are given in (8). Some of them are identities and some of them are products of other generators. These commutators need to be reduced to the independent generators where they can be obtained by using the relation of $B_2(5)$, commutator calculus and Lemma 1-3.

$$\{[a,a^\varphi],[b,b^\varphi],[c,c^\varphi],[l_1,l_1^\varphi],[l_2,l_2^\varphi],[l_3,l_3^\varphi],[l_4,l_4^\varphi],[l_5,l_5^\varphi],[a^{\pm1},(b^\varphi)^{\pm1}],$$
$$[a^{\pm1},(c^\varphi)^{\pm1}],[a^{\pm1},(l^\varphi)^{\pm1}],[a^{\pm1},(l_1^\varphi)^{\pm1}],[a^{\pm1},(l_2^\varphi)^{\pm1}],[a^{\pm1},(l_3^\varphi)^{\pm1}],[a^{\pm1},(l_4^\varphi)^{\pm1}],$$
$$[a^{\pm1},(l_5^\varphi)^{\pm1}],[b^{\pm1},(c^\varphi)^{\pm1}],[b^{\pm1},(l_1^\varphi)^{\pm1}],[b^{\pm1},(l_2^\varphi)^{\pm1}],[b^{\pm1},(l_3^\varphi)^{\pm1}],[b^{\pm1},(l_4^\varphi)^{\pm1}],$$
$$[b^{\pm1},(l_5^\varphi)^{\pm1}],[c^{\pm1},(l_1^\varphi)^{\pm1}],[c^{\pm1},(l_2^\varphi)^{\pm1}],[c^{\pm1},(l_3^\varphi)^{\pm1}],[c^{\pm1},(l_4^\varphi)^{\pm1}],[c^{\pm1},(l_5^\varphi)^{\pm1}],$$
$$[l_1^\pm,(l_2^\varphi)^{\pm1}],[l_1^\pm,(l_3^\varphi)^{\pm1}],[l_1^\pm,(l_4^\varphi)^{\pm1}],[l_1^\pm,(l_5^\varphi)^{\pm1}],[l_2^\pm,(l_3^\varphi)^{\pm1}],[l_2^\pm,(l_4^\varphi)^{\pm1}], \quad (8)$$
$$[l_2^\pm,(l_5^\varphi)^{\pm1}],[l_3^\pm,(l_4^\varphi)^{\pm1}],[l_3^\pm,(l_5^\varphi)^{\pm1}],[l_4^\pm,(l_5^\varphi)^{\pm1}],[a,b^\varphi][b,a^\varphi],[a,c^\varphi][c,a^\varphi],$$
$$[b,c^\varphi][c,b^\varphi],[b,l_1^\varphi][l_1,b^\varphi],[b,l_2^\varphi][l_2,b^\varphi],[b,l_3^\varphi][l_3,b^\varphi],[b,l_4^\varphi][l_4,b^\varphi],$$
$$[b,l_5^\varphi][l_5,b^\varphi],[c,l_1^\varphi][l_1,c^\varphi],[c,l_2^\varphi][l_2,c^\varphi],[c,l_3^\varphi][l_3,c^\varphi],[c,l_4^\varphi][l_4,c^\varphi],$$
$$[c,l_5^\varphi][l_5,c^\varphi],[l_1,l_2^\varphi][l_2,l_1^\varphi],[l_1,l_3^\varphi][l_3,l_1^\varphi],[l_1,l_4^\varphi][l_4,l_1^\varphi],[l_1,l_5^\varphi][l_5,l_1^\varphi],$$
$$[l_2,l_3^\varphi][l_3,l_2^\varphi],[l_2,l_4^\varphi][l_4,l_2^\varphi],[l_2,l_5^\varphi][l_5,l_2^\varphi],[l_3,l_4^\varphi][l_4,l_3^\varphi],[l_3,l_5^\varphi][l_5,l_3^\varphi],$$
$$[l_4,l_5^\varphi][l_5,l_4^\varphi]\}.$$

Lemma 4 leads us to the independent generators of $B_2(5)\otimes B_2(5)$. The proofs of the lemmas are quite lengthy, hence in this paper, we just give some examples of the calculations.

Lemma 4

Let $B_2(5)$ be the second Bieberbach group of dimension five with dihedral point group of order eight.
Then

$$[l_1,l_2^\varphi],[l_1,l_3^\varphi],[l_1,l_4^\varphi],[l_2,l_3^\varphi],[l_2,l_4^\varphi],[l_3,l_3^\varphi],[l_3,l_4^\varphi],[l_3,l_5^\varphi],[a,l_3^\varphi][l_3,a^\varphi],[b,l_3^\varphi][l_3,b^\varphi],[c,l_3^\varphi][l_3,c^\varphi]$$

and all commutator in the form of $[l_i,l_j^\varphi][l_j,l_i^\varphi], i<j$ are equal to the identity.

Proof: By the relation of $B_2(5)$, commutator calculus, Lemma 1-3, we have the following example of calculation:

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$[l_1, l_2^{\varphi}] = [l_1^{-1}, l_2^{\varphi}]^{-1}$	by Lemma 2(i)
$= [c^2, l_2^{\varphi}]^{-1}$	by (2)
$= ([c, l_2^{\varphi}]^2 [[c, l_2], c^{\varphi}])^{-1}$	by (3)
$= ([c, l_2^{\varphi}]^2 [l_2^{\ 2}, c^{\varphi}])^{-1}$	by relation of $B_2(5)$
$= ([c, l_2^{\varphi}]^2 [l_2, c^{\varphi}]^2 [[l_2, c], l_2^{\varphi}])^{-1}$	by (3)
$= ([c, l_2^{\varphi}][l_2, c^{\varphi}])^{-2} [l_2^{-2}, l_2^{\varphi}]^{-1}$	by Lemma 3(iii), relation of $B_2(5)$
$= ([c, c^{\varphi}]^4)^{-2} [l_2, l_2^{\varphi}]^2$	by relation of $B_2(5)$, Lemma 2 (i)
$= [c, c^{\varphi}]^{-8} [c, c^{\varphi}]^8$	since $[l_2, l_2^{\varphi}] = [c, c^{\varphi}]^4$.
$= 1$	

$[l_1, l_4^{\varphi}] = [l_1^{-1}, l_4^{\varphi}]^{-1}$	by Lemma 2(i)
$= [c^2, l_4^{\varphi}]^{-1}$	by (2)
$= ([c, l_4^{\varphi}]^2 [[c, l_4], c^{\varphi}])^{-1}$	by (3)
$= ([c, l_4^{\varphi}]^2 [l_4^{\ 2}, c^{\varphi}])^{-1}$	by relation of $B_2(5)$
$= ([c, l_4^{\varphi}]^2 [l_4, c^{\varphi}]^2 [[l_4, c], l_4^{\varphi}])^{-1}$	by (3)
$= ([c, l_4^{\varphi}][l_4, c^{\varphi}])^{-2} [l_4^{-2}, l_4^{\varphi}]^{-1}$	by Lemma 3(iii), relation of $B_2(5)$
$= ([a, c^{\varphi}][c, a^{\varphi}])^{-4} [l_4, l_4^{\varphi}]^2$	by relation of $B_2(5)$, Lemma 2(i)
$= ([a, a^{\varphi}]^4)^2$	since $o([a, c^{\varphi}][c, a^{\varphi}]) = 4$,
$= [a, a^{\varphi}]^8$	$[l_4, l_4^{\varphi}] = [a, a^{\varphi}]^4$
$= 1$	since $o([a, a^{\varphi}]) = 8$.

Some other commutators in the form $[l_i, l_j],\ i < j$ can be proved equals to identity in similar manner. Hence,
$[l_1, l_2^{\varphi}], [l_1, l_3^{\varphi}], [l_1, l_4^{\varphi}], [l_2, l_3^{\varphi}], [l_2, l_4^{\varphi}], [l_3, l_3^{\varphi}], [l_3, l_4^{\varphi}], [l_3, l_5^{\varphi}], [a, l_3^{\varphi}][l_3, a^{\varphi}], [b, l_3^{\varphi}][l_3, b^{\varphi}],$
$[c, l_3^{\varphi}][l_3, c^{\varphi}]$ are equals to the identity.

Next, the proof of Theorem 1, finding the generators of $B_2(5) \otimes B_2(5)$ is shown.

***Proof Theorem 1**:* The set given in (8) gives the non-independent generators of $[B_2(5), B_2(5)^{\varphi}]$. From the relation of $B_2(5)$, a commutes with l_4, b commutes with l_2 and c commutes with l_1, the commutators $[a, l_4^{\varphi}], [b, l_2^{\varphi}], [c, l_1^{\varphi}], [a, l_4^{\varphi}][l_4, a^{\varphi}], [b, l_2^{\varphi}][l_2, b^{\varphi}]$ and $[c, l_1^{\varphi}][l_1, c^{\varphi}]$ can be eliminated since they can be written as products of other generators as the following computation. By Lemma 2(i) and (ii),

$$[a, l_4^{\varphi}] = [a, l_4^{-\varphi}]^{-1} \qquad [b, l_2^{\varphi}] = [b, l_2^{-\varphi}]^{-1}$$
$$= [a, a^{2\varphi}]^{-1} \qquad = [b, b^{2\varphi}]^{-1}$$
$$= [a, a^{\varphi}]^{-2} \qquad = [b, b^{\varphi}]^{-2}$$

Using similar arguments, $[c,l_1^{\varphi}]=[c,c^{\varphi}]^{-2}$. Furthermore, $[a,l_4^{\varphi}][l_4,a^{\varphi}]=[a,a^{\varphi}]^{-4}$, $[b,l_2^{\varphi}][l_2,b^{\varphi}]=[b,b^{\varphi}]^{-4}$ and $[c,l_1^{\varphi}][l_1,c^{\varphi}]=[c,c^{\varphi}]^{-4}$. Using similarly, $[l_1,l_1^{\varphi}]=[c,c^{\varphi}]^{4}$, $[l_2,l_2^{\varphi}]=[b,b^{\varphi}]^{4}$ and $[l_4,l_4^{\varphi}]=[a,a^{\varphi}]^{4}$.

Meanwhile, all generators in the form of commutators that have negative powers can be eliminated since they can be written as integer powers of its' positive commutators or some other positive commutators. Some examples of hand calculations are shown below:

$$\begin{aligned}
[b,l_1^{-\varphi}] &= [l_1^{-1},[b,l_1]^{\varphi}][b,l_1^{\varphi}]^{-1} && \text{by (7)}\\
&= [l_1^{-1},(l_1^{2})^{\varphi}][b,l_1^{\varphi}]^{-1} && \text{by relation of } B_2(5)\\
&= [l_1,l_1^{\varphi}]^{-2}[b,l_1^{\varphi}]^{-1} && \text{by Lemma 2(i)}\\
&= [c,c^{\varphi}]^{-8}[b,l_1^{\varphi}]^{-1} && \text{since } [l_1,l_1^{\varphi}]=[c,c^{\varphi}]^{4}\\
&= [b,l_1^{\varphi}]^{-1} && \text{since } o([c,c^{\varphi}])=8.
\end{aligned}$$

$$\begin{aligned}
[c,l_2^{-\varphi}] &= [l_2^{-1},[c,l_2]^{\varphi}][c,l_2^{\varphi}]^{-1} && \text{by (7)}\\
&= [l_2^{-1},(l_2^{2})^{\varphi}][c,l_2^{\varphi}]^{-1} && \text{by relation of } B_2(5)\\
&= [l_2,l_2^{\varphi}]^{-2}[c,l_2^{\varphi}]^{-1} && \text{by Lemma 2(i), relation of } B_2(5)\\
&= [c,c^{\varphi}]^{-8}[c,l_2^{\varphi}]^{-1} && \text{since } [l_2,l_2^{\varphi}]=[c,c^{\varphi}]^{4}\\
&= [c,l_2^{\varphi}]^{-1} && \text{since } o([c,c^{\varphi}])=8.
\end{aligned}$$

Some others negative commutators powers can be shown in similar manner.

Now, the set (8) is reduced to the following set:

$$\{[a,a^{\varphi}],[b,b^{\varphi}],[c,c^{\varphi}],[l_5,l_5^{\varphi}],[l_1,l_5^{\varphi}],[l_2,l_5^{\varphi}],[l_4,l_5^{\varphi}],[a,b^{\varphi}],[a,c^{\varphi}],[a,l_1^{\varphi}],[a,l_2^{\varphi}],[a,l_3^{\varphi}],[a,l_5^{\varphi}],$$
$$[b,c^{\varphi}],[b,l_1^{\varphi}],[b,l_3^{\varphi}],[b,l_4^{\varphi}],[b,l_5^{\varphi}],[c,l_2^{\varphi}],[c,l_3^{\varphi}],[c,l_4^{\varphi}],[c,l_5^{\varphi}],[a,b^{\varphi}][b,a^{\varphi}],[a,c^{\varphi}][c,a^{\varphi}],$$
$$[a,l_1^{\varphi}][l_1,a^{\varphi}],[a,l_2^{\varphi}][l_2,a^{\varphi}],[a,l_5^{\varphi}][l_5,a^{\varphi}],[b,c^{\varphi}][c,b^{\varphi}],[b,l_1^{\varphi}][l_1,b^{\varphi}],[b,l_4^{\varphi}][l_4,b^{\varphi}],[b,l_5^{\varphi}][l_5,b^{\varphi}],$$
$$[c,l_2^{\varphi}][l_2,c^{\varphi}],[c,l_4^{\varphi}][l_4,c^{\varphi}],[c,l_5^{\varphi}][l_5,c^{\varphi}]\}.$$

The set (9) is still not independent. Next, the following commutators can be eliminated since they can be written as products of other generators.

$$[b,b^{\varphi}],[l_1,l_5^{\varphi}],[l_2,l_5^{\varphi}],[l_4,l_5^{\varphi}],[a,l_2^{\varphi}],[a,l_3^{\varphi}],[b,c^{\varphi}],[b,l_3^{\varphi}],[b,l_5^{\varphi}],[c,l_2^{\varphi}],[c,l_3^{\varphi}],[c,l_4^{\varphi}],$$
$$[a,b^{\varphi}][b,a^{\varphi}],[a,l_1^{\varphi}][l_1,a^{\varphi}],[a,l_2^{\varphi}][l_2,a^{\varphi}],[b,c^{\varphi}][c,b^{\varphi}],[b,l_1^{\varphi}][l_1,b^{\varphi}],[b,l_4^{\varphi}][l_4,b^{\varphi}],$$
$$[b,l_5^{\varphi}][l_5,b^{\varphi}],[c,l_2^{\varphi}][l_2,c^{\varphi}],[c,l_4^{\varphi}][l_4,c^{\varphi}].$$

Below are some examples of the proof.

$[b,b^{\varphi}]$

$=[b,b^{\varphi}]^{a}$

$=[cl_4^{-1},(cl_4^{-1})^{\varphi}]$ by (2)

$=[c,(cl_4^{-1})^{\varphi}][[c,cl_4^{-1}],l_4^{-\varphi}][l_4^{-1},(cl_4^{-1})^{\varphi}]$ by (3)

$=[c,l_4^{-\varphi}][c,c^{\varphi}][[c,c],l_4^{-\varphi}][l_4^{-2},l_4^{-\varphi}][l_4^{-1},l_4^{-\varphi}][l_4^{-1},c^{\varphi}]$ $[[l_4^{-1},c],l_4^{-\varphi}]$ by (4), relation of $B_2(5)$

$=([c,l_4^{\varphi}][l_4,c^{\varphi}])^{-1}[c,c^{\varphi}][l_4,l_4^{\varphi}]^{2}[l_4,l_4^{\varphi}][l_4^{2},l_4^{-\varphi}]$ by Lemma 3(iii), Lemma 2(i), relation of $B_2(5)$

$=([a,c^{\varphi}][c,a^{\varphi}])^{-2}[c,c^{\varphi}][l_4,l_4^{\varphi}]^{3}[l_4,l_4^{\varphi}]^{-2}$ by relation of $B_2(5)$, Lemma 2(i)

$=([a,c^{\varphi}][c,a^{\varphi}])^{-2}[c,c^{\varphi}][l_4,l_4^{\varphi}]$

$=([a,c^{\varphi}][c,a^{\varphi}])^{-2}[c,c^{\varphi}][a,a^{\varphi}]^{4}.$ since $[l_4,l_4^{\varphi}]=[a,a^{\varphi}]^{4}$.

$[c,l_4^{\varphi}][l_4,c^{\varphi}]=[c,a^{-2\varphi}][a^{-2},c^{\varphi}]$ by (2)

$=([c,a^{\varphi}][a,c^{\varphi}])^{-2}$ by Lemma 3(iii)

$=([a,c^{\varphi}][c,a^{\varphi}])^{2}$ by Lemma 3(iii).

Proving in similar manner, we have $[l_1,l_5^{\varphi}]=[l_2,l_5^{\varphi}]=[l_5,l_5^{\varphi}]^{-2}$,
$[l_4,l_5^{\varphi}]=[a,l_5^{\varphi}]^{-2}$, $[a,l_2^{\varphi}]=[a,l_1^{\varphi}]^{-1}$, $[a,l_3^{\varphi}]=[a,b^{\varphi}]^{-2}[a,c^{\varphi}]^{2}$,
$[b,c^{\varphi}]=[a,a^{\varphi}]^{-6}[c,c^{\varphi}][a,b^{\varphi}]^{-1}[a,c^{\varphi}][a,l_1^{\varphi}]^{-1}$, $[b,1_3^{\varphi}]=[a,a^{\varphi}]^{-4}[c,c^{\varphi}]^{-4}[a,b^{\varphi}]^{-2}[a,c^{\varphi}]^{2}$,
$[c,l_3^{\varphi}]=[b,l_3^{\varphi}]$, $[c,l_4^{\varphi}]=[b,l_4^{\varphi}]$, $[a,b^{\varphi}][b,a^{\varphi}]=[a,a^{\varphi}]^{4}[a,c^{\varphi}][c,a^{\varphi}]$,
$[a,l_1^{\varphi}][l_1,a^{\varphi}]=[a,l_2^{\varphi}][l_2,a^{\varphi}]=[b,l_4^{\varphi}][l_4,b^{\varphi}]=([a,c^{\varphi}][c,a^{\varphi}])^{2}$,
$[b,c^{\varphi}][c,b^{\varphi}]=[c,c^{\varphi}]^{2}([a,c^{\varphi}][c,a^{\varphi}])^{-2}$, $[b,l_1^{\varphi}][l_1,b^{\varphi}]=[c,l_2^{\varphi}][l_2,c^{\varphi}]=[c,c^{\varphi}]^{4}$.
$[b,l_5^{\varphi}][l_5,b^{\varphi}]=[c,l_5^{\varphi}][l_5,c^{\varphi}]$.

With all the elimination, the generating set is reduced to the following set:

$\{[a,a^{\varphi}],[c,c^{\varphi}],[l_5,l_5^{\varphi}],[a,b^{\varphi}],[a,c^{\varphi}],[a,l_1^{\varphi}],[a,l_5^{\varphi}],[b,l_1^{\varphi}],[b,l_4^{\varphi}],[c,l_5^{\varphi}],[a,c^{\varphi}][c,a^{\varphi}],$
$[a,l_5^{\varphi}][l_5,a^{\varphi}],[c,l_5^{\varphi}][l_5,c^{\varphi}]\}$.

Hence, this proves the theorem.

Next, the proof of Theorem 2, the presentation of the nonabelian tensor square of $B_2(5)$ is presented.

Proof Theorem 2. By Theorem 1, $B_2(5)\otimes B_2(5)$ is generated by the set

$\{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9, g_{10}\}$.

Next, the relation of $B_2(5) \otimes B_2(5)$ is determined.

In Wan Nor Farhana et al. (2014b), g_1, g_2 are of order eight, g_3, g_4 are of order four and g_5, g_6 are of order two. Hence we have $g_1^8 = g_2^8 = g_3^4 = 1$.

By Lemma 1 (ii) and (iii), g_1, g_2 and g_3 are central in $\nu(G)$. Hence, $[g_i, g_j] = 1$ for $1 \le i \le 6, 1 \le j \le 13$.

The commutators of the elements of $B_2(5) \otimes B_2(5)$ that are not central of $\nu(G)$ are listed as follows:

$$[g_7,g_{10}],[g_7,g_{12}],[g_8,g_{10}],[g_8,g_{12}],[g_9,g_{10}],[g_9,g_{11}],[g_9,g_{12}],[g_9,g_{13}],[g_{10},g_{11}],$$
$$[g_{10},g_{12}],[g_{10},g_{13}],[g_{11},g_{12}],[g_{11},g_{13}],,[g_7,g_{11}],[g_7,g_{13}],[g_8,g_{13}],[g_8,g_{11}],[g_{12},g_{13}].$$

Now, we compute the above commutators using commutator calculus and Lemma 1 until Lemma 3. Below are some examples of the computation:

$[g_7, g_{10}]$
$= [[a, b^\varphi], [a, l_5^\varphi]]$
$= [[a, b], [a, l_5]^\varphi]$ by Lemma 3(ii)
$= 1$ by Lemma 1(iv).

$[g_7, g_{11}]$
$= [[a, b^\varphi], [b, l_1^\varphi]]$
$= [[a, b], [b, l_1]^\varphi]$ by Lemma 3(ii)
$= [l_4 c^{-1} b, (l_1^2)^\varphi]$ by relation of $B_2(5)$
$= [l_4, l_1^{2\varphi}][[l_4, l_1^2], (c^{-1}b)^\varphi][c^{-1}, l_1^{2\varphi}][[c^{-1}, l_1^2], b^\varphi][b, l_1^{2\varphi}]$ by (3)
$= [l_4, l_1^\varphi]^2 [c^{-1}, l_1^\varphi]^2 [[c^{-1}, l_1], l_1^\varphi][b, l_1^\varphi]^2 [[b, l_1], l_1^\varphi]$ by Lemma 2(i), (6), (4), relation of $B_2(5)$
$= [l_1, l_4^\varphi]^{-2} [c^{-1}, l_1^\varphi]^2 [b, l_1^\varphi]^2 [l_1^2, l_1^\varphi]$ by relation of $B_2(5)$
$= ([c, l_1^\varphi][c, c^\varphi]^4)^2 [b, l_1^\varphi]^2 [l_1, l_1^\varphi]^2$ by Lemma 2(i)
$= [c, l_1^\varphi]^2 [c, c^\varphi]^8 [b, l_1^\varphi]^2 [c, c^\varphi]^8$ since $[l_2, l_2^\varphi] = [c, c^\varphi]^4$
$= ([c, c^\varphi]^{-2})^2 [c, c^\varphi]^{16} [b, l_1^\varphi]^2$
$= [c, c^\varphi]^{12} [b, l_1^\varphi]^2$ since $o([c, c^\varphi]) = 8$
$= [c, c^\varphi]^4 [b, l_1^\varphi]^2$
$= g_2^4 g_{11}^2$.

Proving in similar manner we have
$[g_7,g_8]=[g_7,g_{12}]=[g_8,g_{10}]=[g_8,g_{12}]=[g_9,g_{10}]=[g_9,g_{11}]=[g_9,g_{12}]=[g_9,g_{13}]$
$=[g_{10},g_{11}]=[g_{10},g_{12}]=[g_{10},g_{13}]=[g_{11},g_{12}]=[g_{11},g_{13}]=1, [g_7,g_9]=[g_8,g_9]=g_2^{-4}g_9^2, =[g_8,g_{11}]=g_2^4 g_{11}^2, [g_7,g_{13}]=[g_8,g_{13}]=g_{10}^4,$
$[g_{12},g_{13}]=g_{10}^{-8}$. Hence, it is proved that $B_2(5) \otimes B_2(5)$ has a presentation as in (1).

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CONCLUSION

In this paper, the independent generators and the presentation of the nonabelian tensor square of the second Bieberbach group of dimension five with dihedral point group of order eight, $B_2(5)$ are computed by using the commutator calculus and lemmas. The results can further be used to find other homological functors of $B_2(5)$ such as $J(G)$, and the nonabelian exterior square.

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