ISSN 2462-2052

Implicit Exponentially Fitted Hybrid Method for Special Second Order Initial Value Problems

Kaedah Penyuaian Eksponen Hibrid Tersirat bagi Masalah Nilai Awal Khas Peringkat Kedua

Faieza Samat
Department of Mathematics, Faculty of Science and Mathematics,
Universiti Pendidikan Sultan Idris.
faieza@fsmt.upsi.edu.my

Abstract

An implicit exponentially fitted hybrid method is developed for solving special second order initial value problems. The coefficients of the new method are functions of step-size and the frequency of the problems. The stability region of the method is given. Numerical comparisons on several problems with exponential solutions demonstrate that the new method gives better accuracy compared to the existing method.

Keywords hybrid method, exponentially fitted, second order initial value problems, numerical solution

Abstrak

Satu kaedah penyuaian eksponen hibrid tersirat dibangunkan untuk menyelesaikan masalah nilai awal khas peringkat kedua. Pekali bagi kaedah ini adalah fungsi saiz langkah dan frekuensi bagi masalah. Rantau kestabilan bagi kaedah ini diberikan. Perbandingan berangka pada beberapa masalah dengan penyelesaian eksponen menunjukkan bahawa kaedah baharu memberikan ketepatan yang lebih baik dari kaedah yang sedia ada.

Kata kunci kaedah hibrid, penyuaian eksponen, masalah nilai awal peringkat kedua, penyelesaian berangka

INTRODUCTION

The purpose of this paper is to develop a numerical method for the solution of special second order initial value problems (IVPs)

$$y''(x) = f(x, y(x)),\ y'(x_0) = y'_0,\ y(x_0) = y_0 \qquad 1$$

having exponential solutions. Some authors reduce the second order problems to first order systems of twice dimensions and then solve them using numerical methods designed for first order ordinary differential equations such as in D'Ambrosio and Paternoster (2014). Nevertheless, the development of numerical methods for directly solving these problems is naturally more efficient. To directly solve the second order problems, many papers have

ISSN 2462-2052

been published proposing Runge Kutta Nystrom methods, block methods and multistep methods, see for example Dormand et al. (1987); Fatunla (1990); Lambert and Watson (1976) and Franco (2004). Authors such as Hairer (1979), Cash (1981), Chawla et al. (1986), Fatunla et al. (1999) and Tsitouras (2006) proposed hybrid methods using the ideas underlying the Runge Kutta and multistep methods. In general, the numerical methods are categorized into two classes: 1) Methods with constant coefficients, 2) Methods with variable coefficients. Methods with variable coefficients are useful when accuracy is required in the implementation because the methods are specially adapted to the solutions or to the structure of the problem. In this research, an implicit exponentially fitted hybrid method with variable coefficients is derived. The coefficients of this method are functions of step-size and the frequency of the problem. The accuracy of the new method is measured and compared to the existing method when applied to some problems having exponential solutions.

In this paper, the following class of hybrid methods is considered:

$$Y_i = (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_j f(x_n c_j h, Y_j), \quad i = 1, 2, ..., s \qquad 1.1$$

$$y_{n+1} = 2y_n - y_{n+1} + h^2 \left[\sum_{i=1}^{s} b_i f\left(x_n + c_i h, Y_i\right) \right], \quad x_n = x_0 + nh \qquad 1.2$$

This class of methods has been investigated by Coleman (2003) and can be represented by the Butcher tableau

c	A
	$\mathbf{b}^{\mathrm{T}}$

where $\mathbf{A} = [a_{ij}]$, $\mathbf{b}^{\mathrm{T}} = [b_1, b_2, \ldots, b_s]$ and $\mathbf{c}^{\mathrm{T}} = [c_1, c_2, \ldots, c_s]$. The leading term associated with the local truncation error of a p-th-order hybrid method is given by

$$e_{p+1}(t_i) = \frac{\alpha(t_i)}{(p+1)!}\left[1 + (-1)^{p+2} - \mathrm{b}^T \psi''(t_i)\right], \quad t_i \in T_2, \rho(t_i) = p+2$$

where T_2, $\alpha(t_i)$, $\rho(t_i)$ and $\psi''(t_i)$ are as defined in Coleman (2003). The quantity

$$E = \sqrt{\sum_{i=1}^{n_{p+2}} e_{p+1}^2(t_i)}$$

where n_{p+2} is the number of trees of order $p + 2$, is called the error constant for the p-th order method. In the following sections, we describe the stability analysis for methods with constant and variable coefficients. Then, we derive the exponentially fitted hybrid method. The new method is applied to several second-order problems to provide numerical comparisons with the existing method.

STABILITY ANALYSIS

Methods with constant coefficients

Analysis stability of the methods with constant coefficients is based on the following standard test problem

$$y''(x) = -\lambda^2 y,\ \lambda > 0 \qquad 2$$

whose solution is $y(x) = c_+ e^{i\lambda x} + c_- e^{-i\lambda x}$ where c_+ and c_- are constants. If the hybrid methods [1] solve equation [2], then we will have

$$Y_i = (1+c_i)y_n - c_i y_{n-1} - H^2 \sum_{j=1}^{s} a_{ij} Y_j, \quad i = 1,2,...,s$$

$$y_{n+1} = 2y_n - y_{n-1} - H^2 \sum_{i=1}^{s} b_i Y_i$$

where $H = \lambda h$. The above formula in vector form expression is given as

$$\mathrm{Y} = (\mathrm{e}+\mathrm{c})y_n - cy_{n-1} - H^2 \mathrm{AY} \qquad 3.1$$

$$y_{n+1} = 2y_n - y_{n-1} - H^2 \mathbf{b}^{\mathbf{T}} \mathbf{Y} \qquad 3.2$$

where $\mathbf{e} = (1, 1, \ldots, 1)^{\mathrm{T}}$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_s)^{\mathrm{T}}$. Solving for $\mathbf{Y}$ from equation [3.1] and substituting it into [3.2] gives

$$y_{n+1} - S(H^2)y_n + P(H^2)y_{n-1} = 0$$

where $S(H^2) = 2 - H^2 \mathbf{b}^T (\mathbf{I} + H^2 \mathbf{A})^{-1} (\mathbf{e} + \mathbf{c})$ and $P(H^2) = 1 - H^2 \mathbf{b}^T (\mathbf{I} + H^2 \mathbf{A})^{-1} \mathbf{c}$. The characteristic equation determining the numerical solution of [2] is given by

$$\zeta^2 - S(H^2)\zeta + P(H^2) = 0 \qquad 4$$

The hybrid methods corresponding to the characteristic equation [4] is said to have the interval of stability $(0, H_a)$ if $|P(H^2)| < 1$ and $|S(H^2)| < 1 + P(H^2)$ for all $H \in (0, H_a)$. If $P(H^2) = 1$ and $|S(H^2)| < 2$ for all $H \in (0, H_p)$ then the $(0, H_p)$ is called the interval of periodicity of the hybrid methods.

Methods with variable coefficients

If the hybrid methods defined in [1] have coefficients as a function of one frequency and a step-size, then the interval of stability becomes stability region for one parameter family of methods. Following the stability concept given by Coleman and Ixaru (1996), for exponentially fitted hybrid methods corresponding to the characteristic equation [4], the stability region Ω is a region of the *H*-*v* plane such that $|P(H^2,v)|<1$ and $|S(H^2,v)|<1+P(H^2,v)$.

Derivation of the new method

In this section, we derive a fifth-order diagonally implicit hybrid method with four stages. Then, based on this method, an exponentially fitted hybrid method will be developed. The table of coefficients for the fifth-order diagonally implicit hybrid method is given by

0	0	0	0	0
1	a_{21}	γ	0	0
c_3	a_{31}	a_{32}	γ	0
c_4	a_{41}	a_{42}	a_{43}	γ
	b_1	b_2	b_3	b_4

This method has an algebraic order five and must satisfy the order conditions given in Coleman (2003). Imposing the free parameters to nullify the dissipation error and to minimize the error constant, we obtain

$$\gamma=\frac{1}{30},c_4=-\frac{63}{100},c_3=\frac{23}{37},a_{21}=\frac{29}{30},a_{31}=\frac{281349}{506530},a_{32}=-\frac{12880}{151959},a_{41}=-\frac{87869}{375000},$$
$$a_{42}=\frac{42217}{500000},a_{43}=0,b_1=\frac{1675}{2898},b_2=\frac{31}{13692},b_3=\frac{1874161}{8947092},b_4=\frac{10000000}{47555739}.$$

The diagonally implicit hybrid method is zero dissipative and has phase-lag of order 6. The interval of periodicity is (0, 4.47) and the error constant is $E = 2.55\times10^{-2}$.

Now, we derive the exponentially fitted hybrid method. Assume that $Y_i \approx y(x + c_ih)$. Associating each stage formula in [1] with the linear operator *L*, we have:

$$L_1[y,\mathbf{a}]y(x)=y(x+c_ih)-(1+c_i)y(x)+c_iy(x-h)-h^2\sum_{j=1}^{4}a_j\,f(x+c_ih)$$

$$L_2[y,\mathbf{a}]y(x)=y(x+h)-2y(x)+y(x-h)-h^2\sum_{i=1}^{4}b_if(x+c_ih)$$

According to Raptis and Allison (1978), the linear operator *L* is said to integrate exactly the function $y(x)$ if $L[h, \mathbf{a}]\, y(x) = 0$. In order to derive the new method, firstly we set γ as free parameters a_{22}, a_{33}, a_{44} while other coefficients the same as the diagonally implicit hybrid method derived earlier. Secondly, we impose L_1 to integrate exactly $y(x)=e^{wx}$ to obtain a_{22}, a_{33} and a_{44} . Here, a_{22}, a_{33} and a_{44} are different functions of $v = wh$. Lastly, we impose L_2 to integrate exactly $y(x)=e^{wx}$ and $y(x)=e^{-wx}$, then solve for b_i ($i = 1, \dots 4$) the resulting equations together with the following equations

$$b_1 + b_2 + b_3 + b_4 = 1$$

$$b_1c_1 + b_2c_2 + b_3c_3 + b_4c_4 = 0$$

For small v, the coefficients are subject to heavy cancellations. Therefore, for convenience, the following Taylor series expansions are used:

$$a_{22} = \frac{1}{30} - \frac{1}{30}v^2 - \frac{1}{10}v^2 - \frac{4}{45}v^3 - \frac{11}{240} - \frac{61}{3600}v^5 + \frac{1499}{302400}v^6 - \frac{367}{302400}v^7 + \dots$$

$$a_{33} = \frac{1}{30} + \frac{713}{1519590}v + \frac{7620751}{112449660}v^2 - \frac{422296813}{12481912260}v^3 + \frac{22512293809}{1847323014480}v^4 - \frac{1090316672767}{341754757678800}v^5 + \dots$$

$$a_{44} = \frac{1}{30} + \frac{217}{2000000}v - \frac{54869297}{800000000}v^2 - \frac{3868952009}{600000000000}v^3 - \frac{111988524319}{4800000000000}v^4 - \frac{37944256054829}{4800000000000000}v^5 - \dots$$

$$b_1 = \frac{1675}{2898} - \frac{31}{289800}v^2 - \frac{159219939391}{24994380600000000}v^4 - \frac{233824031181524883}{342173070414000000000000}v^6 - \frac{44696707866647199104216921}{43283387845861178400000000000000000}v^8 + \dots$$

$$b_2 = \frac{31}{13692} - \frac{31}{456400}v^2 + \frac{3662497511}{1686991320000000}v^4 - \frac{108365816285742601}{161664378195600000000000}v^6 + \frac{42113382497089072257309748 7}{2044983251847933936000000000000000000}v^8 - \dots$$

$$b_3 = \frac{1874161}{8947092} + \frac{42439}{298236400}v^2 + \frac{23542761989}{805238280000000}v^4 - \frac{198059327433836899}{771659843724000000000000}v^6 + \frac{245582144044901009275564013}{9761157492777361440000000000000000000}v^8 + \dots$$

J. Sci. Math. Lett. UPSI Volume 1 (2013) 9 - 17
ISSN 2462-2052

$$b_4 = \frac{10000000}{475555739} + \frac{1550}{47555739}v^2 + \frac{529824745097}{16406159286132000}v^4 + \frac{80777951891783011}{22460032062714708000000000}v^6 + \frac{16339888829404728205604481070}{28410952317923346172848000000000000}v^8 - \ldots$$

The stability region for this method is depicted in Figure 1. This shows the usefulness of the new method.

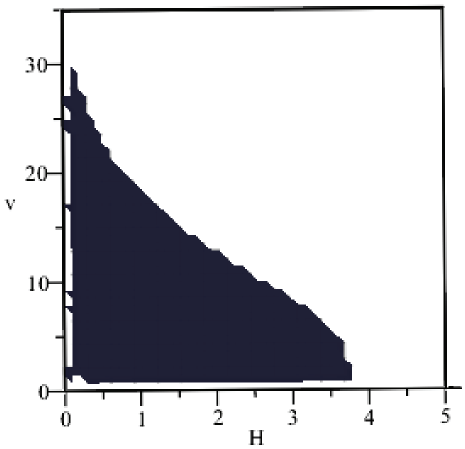

Figure 1 Stability region for the implicit exponentially fitted hybrid method

NUMERICAL EXAMPLES AND DISCUSSIONS

The new method has been applied to several second order problems with exponential solutions to provide numerical evidence of the effectiveness of our method compared to the results of the existing method. Below are the abbreviations of the codes:

EIMH: Implicit exponentially fitted hybrid method with four stages derived in this paper.
FRANCO: Exponentially fitted explicit Runge-Kutta-Nystrom method with four stages derived by Franco (2004).

The numerical results are based on maximum global errors produced by each method for various step-sizes $h = \frac{1}{2^k}$. The maximum global error and its notation are given by

Maximum global error = max $(\|y(x_n) - y_n\|)$
Notation: for example 1.5556E-06 means 1.5556×10^{-6}

Here, $y(x_n)$ and y_n are the exact solution and the numerical solution respectively. Tables 1 to 3 show the maximum global errors of the methods for solving each problem.
The test problems used are as follows.

Problem 1

$$y'' = \lambda^2 y, y(0) = 1, y'(0) = -\lambda, \lambda > 0, x \in [0,1]$$

Solution: $y(x) = \exp(-\lambda x)$
For $\lambda = 5$, $v = -5h$ for EIMH and FRANCO. For $\lambda = 10$, $v = -10h$ for EIMH and FRANCO codes.

Problem 2

$$y'' = y + x - 1, y(0) = 2, y'(0) = -2, x \in [0,5]$$

Solution: $y(x) = 1 - x + \exp(-x)$. The value $v = wh$ for EIMH and FRANCO codes is $v = -h$.

Problem 3

$$y'' + v^2 \left[y - \exp(-\lambda x) \right]^3 = \lambda^2 y, y(0) = 1, y'(0) = -\lambda, x \in [0,5]$$

Solution: $y(x) = \exp(-\lambda x)$. Choose $v = 0.1, \lambda = 0.5$. The value $v = wh$ for EIMH and FRANCO codes is $v = -0.5h$.

Table 1 Maximum global errors of EIMH and FRANCO for Problem 1

λ	*k*	EIMH	FRANCO
5	1	9.136536E-07	4.043165E-01
	2	2.276584E-11	4.383255E-03
	3	4.296043E-15	6.927952E-05
10	3	8.939561E-10	6.504889E-01
	4	1.856364E-13	1.028116E-02
	5	4.724100E-13	2.184213E-04

Table 2 Maximum global errors of EIMH and FRANCO for Problem 2

k	EIMH	FRANCO
2	7.105427E-15	2.377168E-04
3	4.440892E-15	1.118888E-05
4	6.217249E-15	6.058119E-07
5	6.572520E-14	3.522574E-08
6	3.046452E-13	2.123302E-09

Table 3 Maximum global errors of EIMH and FRANCO for Problem 3

k	EIMH	FRANCO
1	3.816392E-16	3.708533E-08
2	2.858824E-15	9.863284E-10
3	2.275957E-15	2.839078E-11
4	2.284284E-14	8.545387E-13
5	6.261658E-14	2.703393E-14

ISSN 2462-2052

From the numerical examples, we can see that the maximum global errors for EIMH are smaller than that of FRANCO code for solving all problems considered. This demonstrates that EIMH gives better accuracy compared to FRANCO code.

CONCLUSIONS

We have derived a new implicit exponentially fitted hybrid method which integrates exactly the second-order problems with exponential solutions. This method compared favorably with the exponentially fitted explicit Runge-Kutta-Nystrom method proposed by Franco (2004). All codes are designed in Microsoft Visual C++ version 6.0 in HP computer with Intel(R)Core(TM)2DuoCPU P8600@2.40GHz.

ACKNOWLEDGEMENT

We acknowledge the support for this research given by the Ministry of Higher Education Malaysia through grant Research Acculturation and Collaboration Effort (RACE).

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