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Quarter-Sweep Iterative Method for Second Kind Linear Fredholm Integral Equations

Kaedah Lelaran Sapuan-Sukuan untuk Persamaan Kamiran Fredholm Linear Jenis Kedua

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Abstract

The main aim of this article is to investigate the application of the quarter-sweep iteration in solving linear Fredholm integral equations of the second kind. The effectiveness of the quarter-sweep iteration concept with Gauss-Seidel iterative method, known as the Quarter-Sweep Gauss-Seidel (QSGS), by using quarter-sweep approximation equation based on quadrature scheme to solve the problem is examined. In addition, the formulation and implementation of the proposed method are also presented. Some numerical simulations are carried out to show that the proposed method is superior compared to the standard method.

Keywords Linear Fredholm equations; Quarter-sweep iteration, Quadrature, Gauss-Seidel

Abstrak

Tujuan utama makalah ini ialah untuk mengkaji aplikasi lelaran sapuan sukuan dalam menyelesaikan masalah persamaan kamiran Fredholm linear jenis kedua. Keberkesanan konsep sapuan sukuan dengan kaedah lelaran Gauss-Seidel, juga dikenali sebagai Gauss-Seidel Sapuan Sukuan (QSGS), dengan menggunakan persamaan penghampiran berdasarkan skema kuadratur untuk menyelesaikan masalah telah dikaji. Sebagai tambahan, formulasi dan pelaksanaan kaedah dicadangkan juga ditunjukkan. Beberapa simulasi berangka juga telah dijalankan untuk menunjukkan kaedah dicadangkan adalah lebih baik jika dibandingkan dengan kaedah piawai.

Kata Kunci Persamaan Fredholm linear, Lelaran sapuan sukuan, Kuadratur, Gauss-Seidel

Introduction

In this article, we consider the numerical solution of Fredholm integral equations of the second kind

$$\mathtt{m}y(x)-\int_{\Gamma}K(x,t)y(t)dt=f(x),\Gamma=[a,b]\mathtt{m}\neq 0 \qquad ...(1)$$

where the parameter λ, kernel $K\in L^2(\Gamma\times\Gamma)$ and free term $f\in L(\Gamma)$ are given, and $y\in L(\Gamma)$ is the unknown function to be determined. The kernel function $K(x,t)$ is assumed to be absolutely integrable and satisfy other properties that are sufficient to imply the Fredholm alternative theorem as mentioned in Theorem 1 and Definition 1. Meanwhile, (1) also can be rewritten in the equivalent operator form

$$(\mathtt{m}-\mathtt{l})y=f. \qquad ...(2)$$

Theorem 1 (Fredholm Alternative) (Atkinson, 1997)
Let $\backslash$ be a Banach space and let $\mathtt{l}:\backslash\rightarrow\backslash$ be compact. Then the equation $(\mathtt{m}-\mathtt{l})y=f,\mathtt{m}\neq 0$ has a unique solution $x\in\backslash$ if and only if the homogeneous equation $(\mathtt{m}-\mathtt{l})z=o$ has only the trivial solution $z=0$. In such a case, the operator $\mathtt{m}-\mathtt{l}:\backslash\overset{1-1}{\underset{onto}{\rightarrow}}\backslash$ has a bounded inverse $(\mathtt{m}-\mathtt{l})^{-1}$.

Definition 1 (Compact operators) (Atkinson, 1997)
Let $\backslash$ and Y be normed vector space and let $\mathtt{l}:\backslash\rightarrow Y$ be linear. Then $\mathtt{l}$ is compact if the set $\{\mathtt{l}\,x|\|x\|x\leq 1\}$ has compact closure in Y. This is equivalent to saying that for every bounded sequence $\{x_n\}\subset\backslash$, the sequences $\{\mathtt{l}x_n\}$ has a subsequence that is convergent to some point in Y. Compact operators are also called completely continuous operators.

In many application areas, numerical approaches were used widely to solve Fredholm integral equations. By solving (2) numerically, we either seek to determine an approximate solution in a chosen finite dimensional space V_n by a projection method (Kaneko, 1989; Chen *et al.*, 2002; Maleknejad & Kajani, 2003; Asady *et al.*, 2005; Kajani & Vencheh, 2005; Xiao *et al.*, 2006; Chen *et al.*, 2007; Long & Nelakanti, 2007; Oladejo *et al.*, 2008)

$$(\mathtt{m}-P_n\mathtt{l})y_n=P_nf \qquad ...(3)$$

where $Y_n\in V_n$ and $P_n:C\rightarrow V_n$ is a projection operator, or use the quadrature method

$$(\mathtt{m}I-\mathtt{l}_n)y_n=f \qquad ...(4)$$

where $\mathtt{l}_n$ approximates $\mathtt{l}$ and is obtained by discretisation of $\mathtt{l}$ by an n point quadrature method; see Laurie (2001), Lin (2003) and, Muthuvalu and Sulaiman (2008; 2009). Such discretisations of integral equations lead to dense linear systems and can be prohibitively expensive to solve as n, the order of the linear system of linear algebraic equations, increases. For large systems, iterative methods are preferred than direct methods because iterative methods often yield a solution within an acceptable error with fewer operations and round-off error are dumped out as the process evolves. Rounding errors due to floating-point arithmetic are frequently become the main problem of direct methods when dealing with large and / or ill conditioned systems (Dias & Leitâo, 1998). For that reason, iterative methods are the natural options for efficient solutions.

The concept of the half-sweep iteration method has been inspired by Abdullah (1991) via the Explicit Decoupled Group (EDG) method to solve two-dimensional Poisson equations. Half-sweep iteration is also known as the complexity reduction approach (Hasan *et al*., 2007) since the implementation of half-sweep iterations will only consider half of all interior node points in a solution domain. Applications of the half-sweep iteration iterative methods have been reviewed in Yousif and Evans (1995), Abdullah and Ali (1996), Othman *et al*. (2000), Sulaiman *et al*. (2004; 2007; 2008) and Abdullah *et al.* (2006).

In 2000, Othman and Abdullah extended this concept by introducing quarter-sweep iterative method via the Modified Explicit Group (MEG) iterative method to solve two-dimensional Poisson equations. Further studies to verify the effectiveness of the quarter-sweep iterative methods have been carried out by Othman and Abdullah (2001), Hasan *et al*. (2005), Sulaiman *et al*. (2004), Hasan *et al*. (2008) and Sulaiman *et al*. (2008). However, in this paper, we examined the applications of the half- and quarter-sweep iteration concepts with Gauss-Seidel (GS) iterative method by using approximation equation based on quadrature scheme for solving problem (1). The standard GS iterative method is also known as the Full-Sweep Gauss-Seidel (FSGS) method. Meanwhile, combinations of the GS method with half- and quarter-sweep iterations are called as Half-Sweep Gauss-Seidel (HSGS) and Quarter-Sweep Gauss-Seidel (QSGS) methods respectively.

The remainder of this paper is organised in following way. In next section, the formulation of the full-, half- and quarter-sweep quadrature approximation equations will be elaborated. The latter sections of this paper will discuss the formulations of the FSGS, HSGS and QSGS iterative methods in solving linear systems generated from discretization of (1) and then some numerical results will be shown to assert the effectiveness of the proposed method. Besides that, analysis on computational complexity is also given and the concluding remarks are given in final section.

Full, Half- and Quarter-sweep Quardrature Approximation Equations

As afore-mentioned, a discretisation scheme based on method of quadrature was used to construct approximation equations for problem (1) by replacing the integral to finite sums. Generally, quadrature method can be defined as follows

$$\int_a^b y(t)dt = \sum_{j=0}^{n} A_j y(t_j) + \text{£}_n(y) \qquad ...(5)$$

where $t_j \ (j = 0,1,2,\cdots,n)$ are the abscissas of the partition points of the integration interval $[a,b]$, $A_j \ (j = 0,1,2,\cdots,n)$ are numerical coefficients that do not depend on the function

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$y(t)$ and $\pounds_n(y)$ is the truncation error of (5). Figure 1 shows the finite grid networks in order to form the full-, half- and quarter-sweep quadrature approximation equations.

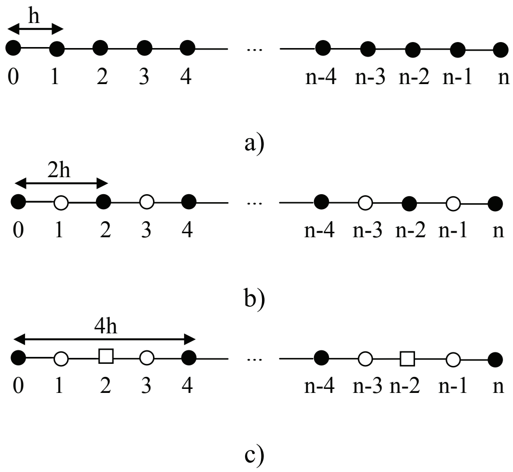

Figure 1 a), b) and c) show distribution of uniform node points for the full-, half- and quarter-sweep cases respectively

Based on Figure 1, the full-, half- and quarter-sweep iterative methods will compute approximate values onto node points of type ● only until the convergence criterion is reached. Then, other approximate solutions at remaining points can be computed using the direct method (Abdullah, 1991; Othman & Abdullah, 2001).

By applying Eq. (5) into Eq. (1) and neglecting the error, $\pounds_n(y)$, a system of linear equations can be formed for approximation values of $y(t)$. The following linear system generated using quadrature method can be easily shown in matrix form as follows

$$M\underset{\sim}{y} = \underset{\sim}{f} \qquad ...(6)$$

where

$$M = \begin{bmatrix} \mathrm{m} - A_0K_{0,0} & -A_pK_{0,2p} & -A_{2p}K_{0,2p} & \Lambda & -A_nK_{0,n} \\ -A_0K_{p,0} & \mathrm{m} - A_pK_{p,p} & -A_{2p}K_{p,2p} & \Lambda & -A_nK_{p,n} \\ -A_0K_{2p,0} & -A_pK_{2p,p} & \mathrm{m} - A_{2p}K_{2p,2p} & \Lambda & -A_nK_{2p,n} \\ M & M & M & O & M \\ -A_0K_{n,0} & -A_pK_{n,p} & -A_{2p}K_{n,2p} & \Lambda & \mathrm{m} - A_nK_{n,n} \end{bmatrix}_{\left(\left(\frac{n}{p}\right)+1\right)x\left(\left(\frac{n}{p}\right)+1\right)}$$

$$\underset{\sim}{y} = \left[\, y_0 \; y_p \; y_{2p} \; \Lambda \; y_{n-2p} \; y_{n-p} \; y_n \right]^T$$

and

$$\underset{\sim}{f} = \left[\, f_0 \; f_p \; f_{2p} \; \Lambda \; f_{n-2p} \; f_{n-p} \; f_n \right]^T$$

In order to facilitate the formulation of the full-, half- and quarter-sweep quadrature approximation equations for problem (1), further discussion will be restricted to repeated trapezoidal (RT) scheme, which is based on linear interpolation formula with equally spaced data. Based on RT scheme, numerical coefficients A_j will satisfy following relation

$$A_j = \begin{cases} \frac{1}{2}ph, & j = 0, n \\ ph, & otherwise \end{cases} \qquad ...(7)$$

where the constant step size, h is defined as follows

$$h = \frac{b-a}{n} \qquad ...(8)$$

and n is the number of subintervals in the interval $[a,b]$. Meanwhile, the value of p, which corresponds to 1, 2 and 4, represents the full-, half- and quarter-sweep cases respectively.

Formulation of the Iterative Methods

As mentioned above, FSGS, HSGS and QSGS iterative methods will be applied to solve linear systems generated from the discretisation of the problem (1), as shown in (6). Let matrix M be decomposed into

$$M = D - L - U \qquad ...(9)$$

where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular matrices respectively. Thus, the general scheme for FSGS, HSGS and QSGS iterative methods can be written as

$$y^{(k+1)} = (D-L)^{-1}\left(U\underset{\sim}{y}^{(k)} + \underset{\sim}{f}\right) \qquad ...(10)$$

Actually, the iterative methods attempts to find a solution to the system of linear equations by repeatedly solving the linear system using approximations to the vector $\underset{\sim}{y}$. Iterations for FSGS, HSGS and QSGS methods continue until the solution is within a predetermined acceptable bound on the error. By determining values of matrices $D, -L$ and $-U$ as stated in (9), the general algorithm for FSGS, HSGS and QSGS iterative methods to solve problem (1) would be generally described in Algorithm 1.

Algorithm 1: FSGS, HSGS and QSGS methods
For $i = 0, p, 2p, \Lambda, n-2p, n-p, n$ and $j = 0, p, 2p, \Lambda, n-2p, n-p, n$

Calculate

$$y_i^{(k+1)} \leftarrow \begin{cases} \dfrac{\left(f_i + \sum_{j=p}^{n} A_j K_{i,j} y_j^{(k)}\right)}{\mathtt{m} - A_i K_{i,i}} & i = 0 \\ \dfrac{\left(f_i + \sum_{j=0}^{n-p} A_j K_{i,j} y_j^{(k+1)}\right)}{\mathtt{m} - A_i K_{i,i}} & i = n \\ \dfrac{\left(f_i + \sum_{j=0}^{i=p} A_j K_{i,j} y_j^{(k+1)} + \sum_{j=i+p}^{n} A_j K_{i,j} y_j^{(k)}\right)}{\mathtt{m} - A_i K_{i,i}} & i = otherwise \end{cases}$$

Numerical Experiences

In order to compare the performances of the iterative methods, several experiments were carried out on the following Fredholm integral equations problems.

Example 1 (Wang, 2006)

Consider the integral equation,

$$y(x) - \int_0^1 (4xt - x^2) y(t) dt = x \qquad ...(11)$$

and the exact solution of problem (11) is given by

$$y(x) = 24x - 9x^2.$$

Example 2 (Polyanin & Manzhirov, 1998)

Consider the integral equation,

$$y(x) - \int_0^1 (x^2 + t^2) y(t).dt = x^6 - 5x^3 + x + 10. \qquad ...(12)$$

Exact solution of problem (12) is

$$y(x) = x^6 - 5x^3 + \frac{1045}{28}x^2 + x + \frac{2141}{84}.$$

There are three parameters considered in numerical comparison such as number of iterations, execution time and maximum absolute error. Throughout the experiments, the convergence test considered the tolerance error, $\mathtt{f} = 10^{-10}$. The experiments were carried out on several different mesh sizes, 513, 1025, 2049, 4097 and 8193. Results of numerical simulations, which were obtained from implementations of the FSGS, HSGS and QSGS iterative methods for Examples 1 and 2, have been recorded in Tables 1 and 2 respectively. Meanwhile, Figures 2 and 3 show execution time versus mesh size for Examples 1 and 2 respectively.

Table 1 Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods (Example 1)

Method	Number of iterations				
	Mest size				
	513	1025	2049	4097	8193
FSGS	194	194	195	195	195
HSGS	193	194	194	195	195
QSGS	192	193	194	194	195
Method	Execution time (seconds)				
	Mesh size				
	513	1025	2049	4097	8193
FSGS	2.62	10.77	38.77	145.01	570.58
HSGS	0.60	2.86	11.24	39.58	155.91
QSGS	0.18	0.58	2.92	12.30	44.96
Method	Maximum absolute error				
	Mesh size				
	513	1025	2049	4097	8193
FSGS	4.69222 E-4	1.17302E-4	2.93249E-5	7.33068E-6	1.83214E-6
HSGS	1.87707E-3	4.69222E-4	1.17302E-4	2.93249E-5	7.33068E-6
QSGS	7.51110E-3	1.87707E-3	4.69222E-4	1.17302E-4	2.93249E-5

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Table 2 Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods (Example 2)

Method	Number of iterations				
	Mest size				
	513	1025	2049	4097	8193
FSGS	194	194	195	195	195
HSGS	193	194	194	195	195
QSGS	192	193	194	194	195
Method	Execution time (seconds)				
	Mesh size				
	513	1025	2049	4097	8193
FSGS	2.62	10.77	38.77	145.01	570.58
HSGS	0.60	2.86	11.24	39.58	155.91
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HSGS	1.87707E-3	4.69222E-4	1.17302E-4	2.93249E-5	7.33068E-6
QSGS	7.51110E-3	1.87707E-3	4.69222E-4	1.17302E-4	2.93249E-5

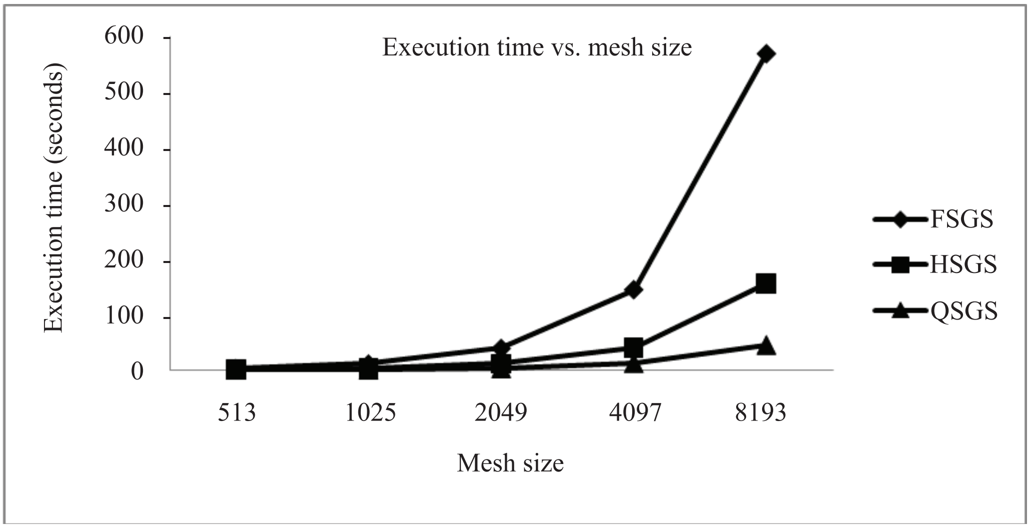

Figure 2 Execution time versus mesh size of the iterative methods for Example 1

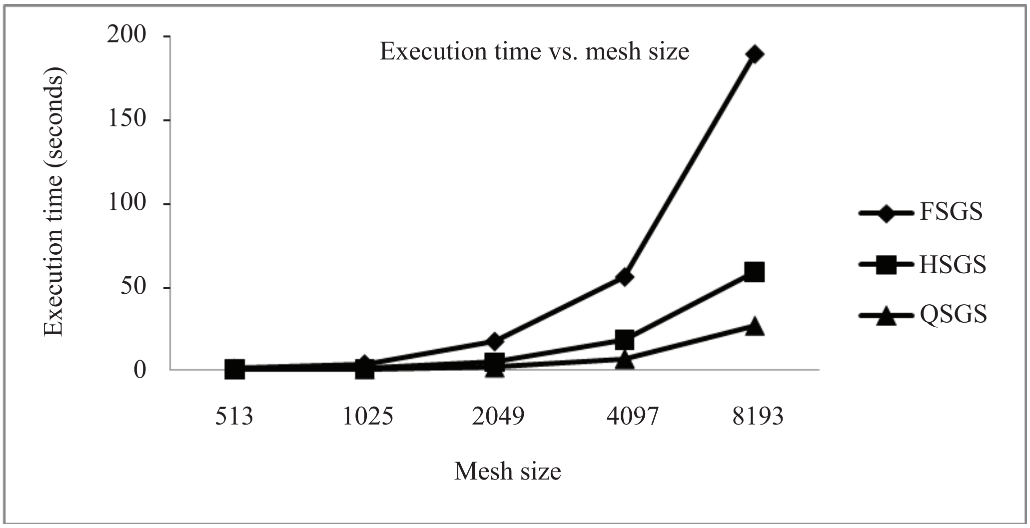

Figure 3 Execution time versus mesh size of the iterative methods for Example 2

Through numerical results obtained for Examples 1 and 2 (refer Tables 1 and 2), it shows that number of iterations for HSGS and QSGS methods are nearly the same compared to the FSGS method. In terms of execution time for both examples, it can be concluded that HSGS and QSGS methods are much faster than FSGS method (refer Tables 1 and 2). Meanwhile, the accuracy of the iterative methods is also in good agreement with QSGS method being the least accurate.

In order to measure the computational complexity of iterative methods, an estimation of the amount of the computational work required has been conducted. The computational work is estimated by considering the arithmetic operations performed per iteration. Based on Algorithm 1, it can be observed that there are $\left(\frac{n}{p}+1\right)$additions/subtractions (ADD/SUB) and $2\left(\frac{n}{p}+1\right)$ multiplications/divisions (MUL/DIV) in computing a value for each node point in the solution domain. From the order of the coefficient matrix, in Eq. (6), the total number of arithmetic operations per iteration for the FSGS, HSGS and QSGS iterative methods has been summarized in Table 3.

Table 3 Total number of arithmetic operations per iteration for FSGS, HSGS and QSGS methods

Method	Arithmetic Operation	
	ADD/SUB	MUL/DIV
FSGS	$(n+1)^2$	$2(n+1)^2$
HSGS	$\left(\frac{n}{2}+1\right)^2$	$2\left(\frac{n}{2}+1\right)^2$
QSGS	$\left(\frac{n}{4}+1\right)^2$	$2\left(\frac{n}{4}+1\right)^2$

Conclusion

In this paper, we present an application of the quarter-sweep iterative method for solving dense linear systems arising from the discretization of the second kind linear Fredholm integral equations by using RT scheme. Overall, the numerical results show that the QSGS method is superior to FSGS and HSGS methods in term of execution time. However, it is not as accurate as FSGS and HSGS iterative methods.

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