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Planar Transition Curves Using Quartic Bezier Spiral

Pembinaan Lengkung Peralihan Satahan menggunakan Perwakilan Lingkaran Kuartik Bezier

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Abstract

A method to construct the transition curves by using a family of the quartic Bezier spiral is described. The applications of quartic spiral discussed are G^2 transition curve joining a straight line and a circle, and joining two straight lines with a pair of spiral segment. A spiral is a curve of monotone increasing or monotone decreasing curvature of one sign. Thus, a spiral cannot have an inflection point or curvature extreme. The family of quartic Bezier spiral form which was introduced has more degrees of freedom and will give a better approximation to clothoid spiral. These methods of constructing transition curves can be simplified by transformation process which extends the application area, and it gives a family of transition curves that allow more flexible curve designs.

Keywords Transition curve, curvature, quartic Bezier spiral

Abstrak

Satu kaedah pembinaan lengkung peralihan yang menggunakan perwakilan lingkaran kuartik Bezier dijelaskan di dalam kertas ini. Perbincangan adalah berkaitan penggunaan lingkaran kuartik untuk menghasilkan lingkaran peralihan dengan darjah keselanjaran G^2 bagi menghubungkan satu garis dengan satu bulatan serta yang menghubungkan dua garis lurus. Lingkaran adalah lengkungan yang memiliki fungsi kelengkungan yang menokok atau menyusut pada satu tandaan, oleh itu ia tidak mempunyai titik lengkokbalas dan juga kelengkungan yang ekstrem. Keluarga lingkaran kuartik yang diperkenalkan ini mempunyai lebih darjah kebebasan dan mampu menjadi penghampiran yang baik terhadap lingkaran clothoid. Kaedah pembinaan lingkaran peralihan ini boleh juga dipermudahkan dengan menggunakan proses tranformasi, ini selanjutnya dapat memperluaskan ruang aplikasinya, serta memberi kita suatu famili lengkungan peralihan yang membenarkan lebih kelenturan dalam rekabentuk lengkungan.

Kata Kunci Peralihan Satahan, Lingkaran Kuartik Bezier

Introduction

Fairness, or smoothness is an important entity of curve and surface, often termed as geometric continuity, G^k or parametric continuity, C^k. A generally accepted mathematical criterion for a curve to be fair is that it should have as few curvature extreme as possible. It is desirable that the curvature extreme occurs only where the designer wants them (Farin, 1997). The fair transition curves which are composed of a single segment or a pair of spiral segments are useful for several Computer Graphics and CAD applications. Applications in which fair curves are of particular importance are the design of horizontal highways, railways route, satellite path, robot trajectories, or aesthetic applications. The importance of this design feature is discussed in (Farin, 1997; Waltan *et al.,* 2003)

Spirals have several advantages of containing neither inflection points, singularities and nor curvature extrema. Such curves are suitable for the construction of the transition curves. One of significant approaches to achieve the transition curve of monotone curvature of constant sign is by using parametric polynomial representation.

The cubic Bezier forms provide a greater range of shapes that allow curve to have cusps, loops, and up to two inflection points. This flexibility makes it suitable for the composition of blending curves. Unfortunately their fairness are not guaranteed. Walton and Meek in has introduced a planar cubic Bezier and PH quintic spiral forms to blend transition curves to produce G^2 order of fairness. Five cases of transition curve have been discussed, namely transition curves between two separate circles; S-shape and C-shape, between two circles in C-oval shape, joining a line and a circle, and between two straight lines. All cases are identified in highway designs as discussed in. More improvements have been done in [8], by increasing the degree of freedom of cubic Bezier spiral. Habib and Sakai in 2005 have also considered a cubic Bezier spiral and suggested the use of tensor parameters for better smoothness and more degrees of freedom. Azhar et al. has introduced a quartic Bezier spiral and proposed a method to construct a G^2 transition curve between two separated circles by composing a pair of spiral segment. Extended studies on quartic Bezier curves give a family of G^3 transition spiral. In this study the function of the lateral change of acceleration (LCA) of the curves is applied, and the comparison on clothoid transition curve is made regarding the vehicle-road dynamics.

In this paper, we propose the construction method of two cases of transition curves; joining a line and a circle, and joining two straight lines. We applied the family of planar quartic Bezier spiral that was presented in (Azhar *et.al.,* 2007). Using quartic forms means more degrees of freedom are obtained. We exploit the extra degrees of freedom to gain the family of transition curves. Although it will involve a long and abstruse mathematical manipulation, the use of symbolic manipulator will make a great help.

The remaining part of this paper is organized as follows. The next section gives a brief discussion of notation and convention. In Section 3, we presented the results of transition curves between a straight line and a circle. A transition spiral joining two straight lines is discussed in Section 4. We derive the necessary condition of the transition in this section.

Notation and convention

The following notations and conventions are used. We consider the dot product of two vectors, A and B is given as $A \bullet B$. The notation of $A \times B$ represents the outer product of two plane vectors A and B [6]. Note that the dot and outer products are $A \bullet B = \| A \| \| B \| Cos\mathrm{i}$ and $A \times B = \| A \| \| B \| Sin\mathrm{i}$, respectively, where i is the turning angle from A to B. Positive angle is measured anti-clockwise. The norm or length of a vector A is $\|A\|$. In this paper we denote $R(t)$ as a quartic curve. If T is the unit tangent vector to $\dot{R}(t)$ at t, then $T = R(t)/\|R(t)\|$ and $\|T\| = 1$. The unit normal vector N at $R(t)$ is perpendicular to T and the angle measured anti-clockwise from T is $\mathrm{r}/2$. The signed curvature of a plane curve $R(t)$ is

$$\mathrm{l}(t) = \frac{R'(t) \times R''(t)}{\| R'(t) \|^3} \qquad ...(1)$$

The signed radius is the reciprocal of (1). It is known that $\mathrm{l}(t)$ is positive sign when the curve segment bends to the left and its negative sign if it bends to the right at t. $R'(t)$ and $R''(t)$ are first and second derivatives of $R(t)$.

The quartic Bezier spiral is defined in the following theorem. First, we consider a standard quartic Bezier curve

$$R(t) = \sum_{i=0}^{4} P_i \binom{4}{i} (1-t)^{4-i} t^i, \quad 0 \le t \le 1, \qquad ...(2)$$

where P_i, $i = 0,1,2,3,4$ are the control points. The following theorem defines the necessary condition of a single quartic Bezier spiral of monotonically increasing.

Theorem 1

Given a beginning point, P_0, and two unit tangent vectors T_0 and T_1 at beginning and ending points, respectively. Let i be the anti-clockwise angle from T_0 to T_1, where $0 \le \mathrm{i} \le \frac{\mathrm{r}}{2}$. While the curvature of the ending point is given as $1/r$, it is assumed that centre of the circle of curvature at ending point is to the left of the direction of and in positive value, i.e., $r > 0$. If the control points are given as

$$P_1 = P_0 - \frac{r \mathrm{t}_0 \mathrm{p}^2 Sec\mathrm{i}\, Tan\mathrm{i}}{108 \mathrm{a}_0^3 (\mathrm{t}_0 - 1) \mathrm{t}_1^2} T_0$$

$$P_2 = P_1 - \frac{r \mathrm{p}^2 Sec\mathrm{i}\, Tan\mathrm{i}}{108 \mathrm{a}_0^3 \mathrm{t}_1^2} T_0$$

$$P_3 = P_2 - \frac{r \mathrm{p} [-9 \mathrm{a}_0 Cos\mathrm{i} (-1 + \mathrm{t}_1) T_1 + \mathrm{p} T_0] Sec\mathrm{i}\, Tan\mathrm{i}}{108 \mathrm{a}_0^2 \mathrm{t}_1^2}$$

$$P_4 = P_3 + \frac{r \mathrm{p}\, Tan\mathrm{i}}{12 \mathrm{a}_0 \mathrm{t}_1^2} T_1, \qquad ...(3)$$

where

$$\mathrm{p} = (2\mathrm{t}_1 + 3\mathrm{a}_0 (4\mathrm{t}_1 - 3)), \qquad ...(4)$$

and a_0, t_0, t_1 are positive arbitraries such that,

$$t_1 = \frac{9}{14},\ 0 < a_0 \leq \frac{8}{25},\ t_0 = \frac{15(1 - a_0 - t_1 + a_0 t_1)}{27 - 15a_0 - 26t_1 + 15a_0 t_1} \qquad ...(5)$$

then $R(t)$ as defined by (Azhar. *et al.,* 2007) is a spiral with increasing curvature.

The proof of Theorem 1 can be referred to [2]. This quartic Bezier spiral has all the properties of ordinary quartic Bezier, and in addition $l(0) = 0$, $l(1) = 1/r$, $l'(1) = 0$, $l''(0) = 0$ and $l'(t) \neq 0$ for $0 < t < 1$. Note that this quartic Bezier spiral has six degrees of freedom; one for each of i, r, a_0, T_0 and two for P_0, where t_0 is depending on a_0 and t_1, while the unit vector T_1 is determined by i and T_1.

Transition spiral joining a straight line and a circle

The following theorem gives the result for the transition spiral from a straight line to a circle. For defining the necessary condition of this transition, we use the intermediate value theorem in order to obtain a unique solution.

Theorem 2 Given a point, A_0, a unit vector, T_0, and a circle Ω_0 of radius $r > 0$ centered at C_0 where $T_0 \times (C_0 - A_0) > 0$. Let L be the straight line through A_0 and parallel to T_0, and let d be the perpendicular distance from C_0 to L. If $d > r$, then there is a unique quartic Bezier spiral $R(t)$ as defined in Theorem 1 that joins L and X_0 such that the points of contact are G^2 continuity.

Proof From Theorem 1 and Figure 1, we can write $R(1)$ in the following terms

$$R(1) \equiv P_4 + P_0 = a_0 T_0 + b_0 T_0 = P_0 + v_0 T_0 + G_0 - rN_1 \qquad ...(6)$$

where $G_0 = C_0 - A_0$ and a_0, b_0 are given by

$$a = -\frac{r p^2 \, Sec\, i \; Tan\, i}{108 a_0^3 (-1 + t_0) t_1^2} \qquad ...(7)$$

$$b_0 = \frac{r p \; Tan\, i}{12 a_0 t_1^2}. \qquad ...(8)$$

Let N_0 and N_1 as unit vectors normal to T_0 and T_1, respectively. From Figure 1, we have

$$T_0 \bullet T_1 = Cos\, i,\ T_0 \bullet N_1 = -Sin\, i,\ N_0 \bullet T_1 = Sin\, i,\ N_0 \bullet N_1 = Cos\, i \qquad ...(9)$$

Taking the dot product of (6) with T_0 and N_0, and from (9), respectively, and after some simplifications, yield

$$G_0 \bullet T_0 = a_0 - \mathrm{v}_0 + b_0 Cos\,\mathrm{i} - rSin\,\mathrm{i}, \qquad ...(10)$$

$$G_0 \bullet N_0 = b_0 Sin\,\mathrm{i} + rCos\,\mathrm{i}. \qquad ...(11)$$

We let $d = G_0 \bullet N_0$, substituting a_0, b_0 from (7)-(8), and $\mathrm{t}_0, \mathrm{t}_1$ from Theorem 1 into (11), hence we have the following result,

$$d = rCos\,\mathrm{i} + \frac{49r\mathrm{p}Sin\,\mathrm{i}\;Tan\,\mathrm{i}}{243\mathrm{a}_0} \qquad ...(12)$$

Equation (12) has a unique solution for $0 < \mathrm{i} < \frac{\mathrm{h}}{2}$ and $0 < \mathrm{a}_0 < \frac{8}{25}$. The analysis is as follows.

First, we represent $0 < \mathrm{a}_0 < \frac{8}{25}$ by $\mathrm{a}_0 = \frac{\mathrm{h}}{1+\mathrm{h}}\frac{8}{25}$, $\mathrm{h} \geq 0$, and let $\Lambda = Tan\,\mathrm{i}$. Hence,

after substituting a_0 and p into (12), we obtain

$$q(\Lambda, \mathrm{h}) = d - \frac{(216\mathrm{h} + 7\Lambda^2(25 + 17\mathrm{h}))r}{216\mathrm{h}\sqrt{1+\Lambda^2}} = 0 \qquad ...(13)$$

Next, the first derivative of $q(\Lambda, \mathrm{h})$with respect to Λ is

$$\frac{dq(\Lambda, \mathrm{h})}{d\Lambda} = -\frac{\Lambda r(350 + 22\mathrm{h} + 7\Lambda^2(25 + 17\mathrm{h}))}{216\mathrm{h}(1+\Lambda^2)^{3/2}} \qquad ...(14)$$

Since

$\lim_{\Lambda \to 0^+} q(\Lambda, \mathrm{h}) = d - r > 0$, if $d > r$,

$$\lim_{\Lambda \to \infty} q(\Lambda, \mathrm{h}) < 0 \text{ and } \frac{dq}{d\Lambda} < 0,\ \forall \Lambda \in (0, \infty) \qquad ...(15)$$

We therefore concluded that (12) has a unique solution for $0 < \mathrm{i} < \frac{\mathrm{r}}{2}$, and $0 \leq \mathrm{a}_0 \leq \frac{8}{25}$ that if $d > r$. This completes the proof of Theorem 2.

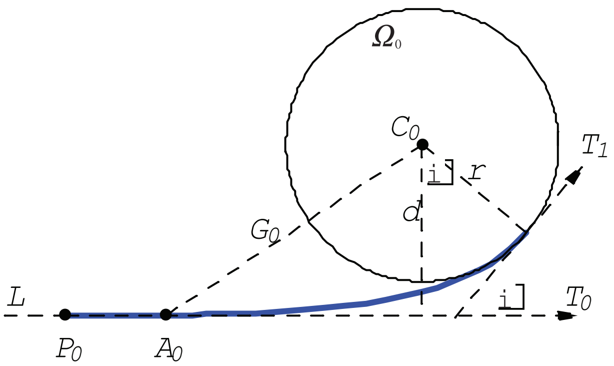

Figure 1 Transition spiral between a straight line and a circle

So a transition spiral from a straight line to a circle can be obtained by assigning a value of $\mathrm{i} \in (0, \mathrm{r}/2)$ in (12) after we compute d from the given data. Hence, we obtain a_0, and v_0 is computed from (10). Next, we compute P_0 from $A_0 - P_0 = \mathrm{v}_0 T_0$, and T_1 is obtained from T_0 and i.

Transition spiral joining two straight lines

The following theorem discusses the necessary condition for the existence of a transition curve joining two straight lines with G^2 contacts. The transition curve is a composition of a pair of quartic spiral as defined by Theorem 1.

Theorem 3 Given three points Q_0, Q_1, and Q_2. Let L_1 be the directed lines from Q_1 to Q_0, and let L_2 be the directed lines from Q_2 to Q_0. Let $\} \in (0, \mathrm{r})$ be the angle at Q_0 formed by Q_1, Q_0, and Q_2. For any value $r > 0$ and $\mathrm{i} = (\mathrm{r} - \})/2$,

if $\ell > \dfrac{(335 + 97Cos2\mathrm{i})r}{216(1 + Cos2\mathrm{i})}$, then the pair of quartic Bezier spiral as defined in (2)-(5) can be joined symmetrically with respect to $C_0 - Q_0$, which emanating from L_1 and L_2, and meeting at $P_{4,1} = P_{4,2}$, such that all points of contact are G^2, where $\ell = \| C_0 - Q_0 \|$ and C_0 is the centre of osculating circle of radius r.

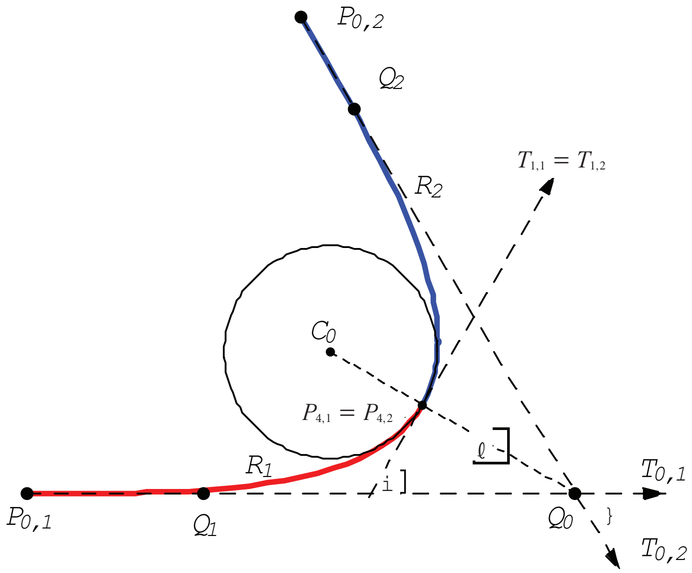

Figure 2 Transition spiral between two straight lines

Proof Let us define $T_{0,1}$, $T_{1,1}$ and $T_{0,2}$, $T_{1,2}$ be the unit tangent vectors of beginning and ending point of segment $R_1(t)$ and $R_2(t)$, respectively. According to Figure 2, we define

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$$T_{0,1} = \frac{(Q_0 - Q_1)}{\| Q_0 - Q_1 \|}, \text{ and } T_{0,2} = \frac{(Q_0 - Q_2)}{\| Q_0 - Q_2 \|} \qquad ...(16)$$

The angle between two directed lines, $\}$, is obtained from $T_{0.1} \bullet T_{0.2} = Cos\}$. Since the two spirals are joined symmetrically, these imply $T_{0,1} \bullet T_{1,1} = T_{0,2} \bullet T_{1,2} = Cos\mathtt{i}$. Without lose of generality, we only consider segment $R_1(t)$ to obtain the necessary condition of this transition. From Theorem 1 and Figure 2, we get

$$R_1(1) \equiv P_{0,1} + a_0 T_{0,1} + b_0 T_{1,1} \equiv P_{0,1} + \mathtt{v}_1 T_{0,1} + \ell N_{1,1} - r N_{1,1} \qquad ...(17)$$

where a_0 and b_0 are as (7)-(8). $N_{1,1}$ is a unit normal vector of $T_{1,1}$. Let $N_{0,1}$ be the unit normal vector of $T_{0,1}$, so we set the following dot products,

$$T_{0,1} \bullet T_{1,1} = Cos\mathtt{i}, T_{0,1} \bullet N_{1,1} = -Sin\mathtt{i}, N_{0,1} \bullet N_{1,1} = Cos\mathtt{i} \qquad ...(18)$$

By using (18), and taking the dot products of (17) with $T_{0,1}$ and $N_{0,1}$, respectively, we have

$$(a_0 - \mathtt{v}_1) + b_0 Cos\mathtt{i} + (\ell - r) Sin\mathtt{i} = 0 \qquad ...(19)$$

$$(\ell - r) Cos\mathtt{i} = b_0 Sin\mathtt{i} \qquad ...(20)$$

To simplify the analysis, we represent $0 < \mathtt{a}_0 \leq \frac{8}{25}$ by $\mathtt{a}_0 = \left(\frac{\mathtt{h}}{1+\mathtt{h}}\right)\frac{8}{25}, \mathtt{h} \geq 0.$

Next, after substituting $b_1, \mathtt{p}, \mathtt{t}_1$ and followed by $\mathtt{a}_0$, into (20), we obtain

$$(\ell - r) Cos\mathtt{i} = \frac{7(25 + 17\mathtt{h}) r Sin\mathtt{i}\, Tan\mathtt{i}}{216\mathtt{h}} \qquad ...(21)$$

From (21), let us write

$$q(\mathtt{h}, \mathtt{i}) = (\ell - r) Cos\mathtt{i} - \frac{7(25 + 17\mathtt{h}) r Sin\mathtt{i}\, Tan\mathtt{i}}{216\mathtt{h}} \qquad ...(22)$$

Differentiation of (22) with respect to $\mathtt{h}$ is

$$q'(\mathtt{h}, \mathtt{i}) = \frac{175 r Sin\mathtt{i}\, Tan\mathtt{i}}{216\mathtt{h}^2} \qquad ...(23)$$

It is clear that

$$\lim_{\mathtt{h} \to 0} q(\mathtt{h}, \mathtt{i}) < 0,$$

$$\lim_{\mathtt{h} \to \infty} q(\mathtt{h}, \mathtt{i}) = -\frac{1}{432}(335r - 216\ell + (97r - 216\ell) Cos2\mathtt{i}) Sec\mathtt{i} > 0, \text{ if}$$

$$\ell > \frac{(335 + 97Cos2\mathtt{i})r}{216(1 + Cos2\mathtt{i})},$$

$$\frac{dq(\mathtt{h},\mathtt{i})}{d\mathtt{h}} > 0 \text{ for } 0 < \mathtt{h} < \infty \qquad \text{...(24)}$$

This conclude that (21) has a unique solution for $0 < \mathtt{a}_0 \leq \frac{8}{25}$ when

$$\ell > \frac{(335 + 97Cos2\mathtt{i})r}{216(1 + Cos2\mathtt{i})}.$$

After substituting a_1 and b_1 into (19), we can obtain $\mathtt{v}_1$ for placing $P_{0,1}$ which comes from equation $Q_0 - P_{0,1} = \mathtt{v}_1 T_{0,1}$. Then follows by the second segment $R_2(t)$ which can be drawn in the same manner.

Summary

It has been demonstrated that fair curves can be designed interactively using quartic Bezier spirals. Since quartic Bezier also has NURBS representations, curves designed using a combination of quadratic, cubic, quartic spirals, circular arcs and the straight line segment can be represented entirely by NURBS. Generally, this quartic Bezier spiral is more flexible than clothoid because it has six degrees of freedom, and hence will give a family of transition curves in all cases discussed earlier.

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