RESEARCH PAPER

Extrapolation of General Linear Methods with Inherent Runge-Kutta Stability

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Abstract

General linear methods have been proven to be very efficient in solving stiff and non-stiff differential equations. Extrapolation is proven to increase the accuracy of any methods. This paper investigates the accuracy and efficiencies of explicit general linear methods with inherent Runge-Kutta stability (IRKs) with and without extrapolation. The numerical results on the Van der Pol (VDP) and Brusselator (Bruss) non-stiff test equations showed that IRKs with extrapolation are more efficient and accurate than itself without extrapolation.

Keywords: Inherent Runge-Kutta stability, IRKs, Extrapolation, General linear methods.

INTRODUCTION

The ordinary differential equations are considered by

$$y' = f(y(t)), \quad t \in [t_0, t_n], \; y(t_0) = y_0. \tag{1}$$

On the following uniform grid

$$x_n = x_0 + nh, \; Nh = X - x_0, \; n = 0,1,\dots,N,$$

then the general linear methods can be solve Eq. 1, which formulated by

$$Y_i^{[n]} = \sum_{j=1}^{s} a_{ij} h f\left(Y_j^{[n]}\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]},$$

$$y_i^{[n]} = \sum_{j=1}^{s} b_{ij} h f\left(Y_j^{[n]}\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]}, \tag{2}$$

where $Y_i^n, i = 1,2, \dots, s$ denotes as internal stages of methods, $y_j^{[n-1]}$ is given as the incoming quantities through the step $n-1$, $y_j^{[n]}$ is the outgoing quantities through the step number n, $f\left(Y_j^{[n]}\right)$ is the corresponding stage derivatives. However, the internal stages can be approximating of stage order q as

$$Y_j = y\left(x_{n-1} + c_j h\right) + O(h^{q+1}), \qquad j = 1,2, \dots, s,$$

and the outgoing quantities can be approximating of order p as

$$y_j^{[n]} = \sum_{k=0}^{p} \alpha_{jk} h^k y^{(k)}(x_n) + O(h^{p+1}), \qquad j = 1,2, \dots, s.$$

In this paper, the idea of extrapolation is applied with general linear methods that have inherent Runge-Kutta stability (IRKS) (Wright, 2002). Extrapolation technique is based on the idea by Richardson in 1927 (Richardson, 1927) where he used two of approximations (h_0, h_1) to accelerate the convergence of a sequence. His idea had been applied with some efficient numerical solutions, such as Runge-Kutta (RK) methods (Gorgey, 2012; Ismail, 2013).

General linear methods are divided into four types, the type I known as the explicit methods to solve the non stiff differential equations while the type II known as the implicit methods considered to solve the stiff differential equations. These two types are considered in sequential computations. The other two types are given in parallel computations (Butcher, 2016). Since the explicit methods proved have lower cost implementations (Jackiewicz, 2009) then we are interested in the first type (explicit methods) with using the extrapolation technique to improve the accuracy to solve the ordinary differential equation.

Many researchers have been trying in many ways to construct an efficient general linear methods to solve the ordinary differential equations. One of them is Bazuaye (2017). In his research, he extended exponential general linear methods with initial value problem to solve the ordinary differential equations. He used the feature feature of exponential, that it allows to derive the order conditions of general linear methods, which in turn assisted in the construction of a family of methods of higher order. Another example of using the effective of general linear methods for solving the methods, is explained by Mahdi (2018). In his research, he used the general linear methods to solve the volterra integro-differential equations. He proved general linear methods is efficient in solving the non-linear volterra integro-differential. Farzi (2018) developed and generalized the scheme of Adams scheme which is known as Fuzzy general linear methods for solving fuzzy differential problems under the hardly generalized differentiability. He showed that the order of accuracy is more efficient by the novel scheme.

The article is organized as follows. The first section explains the formulation of general linear methods and its coefficient matrices. The constructions of IRKS methods is proposed in Section 3. The deriving of the order conditions are given in Section 4. Section 5 reviews the assumptions and requirements of applying the extrapolation with the current methods. Numerical results for VDP and Bruss test equations are presented in Section 6. The final section explains the conclusions and future works.

GENERAL LINEAR METHODS

Ordinary differential equations can be solved approximately by numerical methods, given by Runge-Kutta and linear multi-step methods. However, these methods have some disadvantages such as, high computational cost in Runge-Kutta methods and weak stability in multi-step methods. Forty years ago, Butcher in Butcher (2009) introduced a method upgrade the disadvantage in the previous methods to solve the ordinary differential equations. These methods known as general linear methods, which approved have a good balanced among the accuracy and stability within reducer cost.

For convenience, the general linear methods Eq. 2 can be represented by

$$Y^{([n])} = h(A \otimes I_m)F(Y^{[n]}) + (U \otimes I_m)y^{[n-1]},$$

$$y^{([n])} = h(B \otimes I_m)F\left(Y^{[n]}\right) + (V \otimes I_m)y^{[n-1]}.$$

The coefficients matrix of general linear methods assumed as $a_{ij} = A, u_{ij} = U, b_{ij} = B$ and $v_{ij} = V$. General linear methods classified into four types according to the way of structure the coefficient matrix A, which is given by

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 \\ a_{21} & \lambda & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ a_{s1} & a_{s2} & \cdots & \lambda \end{bmatrix},$$

where if $\lambda = 0$ then methods are known as explicit methods, otherwise then methods known as implicit methods, these types are given in sequential computations. The other two types given in parallel computations. Furthermore, the coefficient matrix A determines the cost of implementation, which is considered here to these methods as Runge-Kutta methods to make the cost lower.

The coefficient matrix V explains the stability of general linear methods, which is given a matrix of rank one to ensure zero stability. To approve this assumption, then coefficient V has selected as a simple structure, which the first column was given to equal basis vector e; that's mean Ve=e (Jackiewicz, 2009).

IRKS Methods

General linear methods with inherent Runge-Kutta stability (IRKs) have been studied by Wright (2002), known as IRKs methods. The restrictions of IRKs assumed on the way of formulating the coefficients A, U, B and V of general linear methods to make sure their stability is similar to the stability of the Runge-Kutta method.

The coefficient V considered as $Ve = e$, and then general linear methods have inherent Runge-Kutta stability if

$$BA = XB, \tag{3}$$

$$BU \equiv XV - VX, \tag{4}$$

for some matrix X. The following definition shows how the stability properties of Runge-Kutta methods assumed to IRKs methods.

Definition: General linear methods are said to have Runge-Kutta stability, then the stability polynomial given by

$$\mathrm{P}(\omega) = \det\big(\omega \mathrm{I} - \mathrm{M}(z)\big) = \omega^{r-1}\big(\omega - \mathrm{R}(z)\big),$$

where $\mathrm{R}(z)$ denotes the stability function of Runge-Kutta methods, as well as $\mathrm{M}(z)$ is defined as the stability matrix of general linear methods, which is supposed by

$$M(z) = V + zB(I - zA)^{-1}U.$$

Since our aim in this paper concentrate on the explicit part, then the stability function $R(z)$ considered as same as explicit Runge-Kutta methods, given by

$$R(z) = R_p(z;\eta)1 + z + \frac{z^2}{2!} + \cdots + \frac{z^p}{p!} + \eta\frac{z^{p+1}}{(p+1)!},$$

here η is error constant and the order condition is given by p, explained by details next.

Order Conditions

The construction and implementations of general linear methods are completely tough, to make them easier if assuming the order of method p is equal to stage order q. However, before starting deriving the order condition, consider the input and output approximations firstly are given in Nordsieck form as follows:

$$y^{[n]} \approx \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ \vdots \\ h^p y^{(p)}(x_n) \end{bmatrix} + O(h^{p+1}),$$

Secondly, by following Wright (2002) the stage values Y^n and the output approximations $y^{[n]}$ given by,

$$Y^n = Cy^{[n-1]} + O(h^{p+1}),$$
$$y^{[n]} = Ey^{[n-1]} + O(h^{p+1}),$$

where

$$C=\begin{bmatrix}1 & c_1 & \frac{c_1^2}{2!} & \cdots & \frac{c_1^{(p-1)}}{(p-1)!} & \frac{c_1^p}{p!}\\ 1 & c_2 & \frac{c_2^2}{2!} & \cdots & \frac{c_2^{(p-1)}}{(p-1)!} & \frac{c_2^p}{p!}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 1 & c_p & \frac{c_p^2}{2!} & \cdots & \frac{c_p^{(p-1)}}{(p-1)!} & \frac{c_p^p}{p!}\\ 1 & c_{p+1} & \frac{c_{p+1}^2}{2!} & \cdots & \frac{c_{p+1}^{(p-1)}}{(p-1)!} & \frac{c_{p+1}^p}{p!}\end{bmatrix},$$

$$E=\exp(K)=\begin{bmatrix}1 & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{(p-1)!} & \frac{1}{p!}\\ 0 & 1 & \frac{1}{1!} & \cdots & \frac{1}{(p-2)!} & \frac{1}{(p-1)!}\\ 0 & 0 & 1 & \cdots & \frac{1}{(p-3)!} & \frac{1}{(p-2)!}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{1!} & \frac{1}{2!}\\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{1!}\\ 0 & 0 & 0 & \cdots & 0 & 1\end{bmatrix}$$

Using these assumptions in the form of general linear methods Eq. 2 and using functions of complex variables, it is useful in the following theorem, explaining the order conditions.

Theorem: General linear methods in the Nordsieck representation have the condition $\mathrm{p}=\mathrm{q}$ iff

$$\exp(\mathrm{cz})=\mathrm{zAexp}(\mathrm{cz})+\mathrm{UZ}+\mathrm{O}(\mathrm{z}^{\mathrm{p}+1}), \tag{5}$$

$$\exp(\mathrm{z})\mathrm{Z}=\mathrm{zBexp}(\mathrm{cz})+\mathrm{VZ}+\mathrm{O}(\mathrm{z}^{\mathrm{p}+1}), \tag{6}$$

where $\exp(\mathrm{cz})$ denotes vector of components and Z defined by

$$\mathrm{Z}=[1\ \ \mathrm{z}\ \ \mathrm{z}^2\ \ \ldots\ \mathrm{z}^{\mathrm{p}-1}\ \ \mathrm{z}^{\mathrm{p}}],$$

We can conclude from the above theorem, the following results:

$$U=C-ACK, \tag{7}$$

$$V=E-BCK. \tag{8}$$

These results help us to know the way to construct the order of IRKs methods. That means, if the coefficients A and B are constructed in Eq. 3, then it's easy to know the other coefficient matrices U and V by results Eq. 7 and Eq. 8.

Applications of Extrapolation

Extrapolation technique is one of the efficient numerical procedures that can be utilized efficiently in the efforts of some programs to enhance the performance by which long time-dependent engineering and scientific issues are dealing with the computers. Richardson first derived the extrapolation (Richardson, 1911). It is an approximation method in the numerical solution of differential equations. The different phenomena in engineering and science are successfully described using some advanced large-scale mathematical models. The extrapolation in most applications is used as an initial technique to evaluate the magnitude of the computational errors and is used to enhance the given results' accuracy (Richardson, 1927).

There are two ways to apply extrapolation. The active case occurs if the extrapolated value is used to compute the next iterations. On the other hand, if the extrapolation is only applied without using the extrapolated solution, the approach is known as passive.

The extrapolation technique can be performed with the general linear methods by using the assumptions (Zlatev, 2018) in the following steps:

- Implement one large step N with the step-size h by applying $y_{n-1.}$ as a starting values to compute:

$$z_n = R(z)y_{n-1.}$$

- Implement two small steps with the step-size $1/2h$ to evaluate the approximation $\boldsymbol{w_n}$ of $y(t_n)$.

$$w_n = \left[R\left(\frac{z}{2}\right)\right]^2 y_{n-1.}$$

- Evaluate y_n such as

$$y_n = \frac{2^p \boldsymbol{w_n} - \boldsymbol{z_n}}{2^p - 1} = \frac{2^p \left[R\left(\frac{z}{2}\right)\right]^2 - R(z)}{2^p - 1} y_{n-1}$$

NUMERICAL RESULTS

The numerical experiments of the current methods are used the extrapolation technique in their implementations. These experiments are based on solving some efficient test equations such as Van der Pol (VDP) and Brusselator (Bruss) non-stiff test equations, which is found in Hairer (1993). The numerical results of these methods are presented in Figures using the MATLAB software. The MATLAB code constructs IRKs methods based on irks14t, which is considered by Abdi (2019). Since we are using the extrapolation technique with the current methods, then the given code is updated to a couple of codes such as irks2xactive and irks2xpassive. They explain extrapolated IRKS of order two in active and passive modes. The global errors on all the graphs are computed by taking the maximum norm such that for different tolerances on which the tolerance is reduced by 1/10 at every iteration. The starting tolerance used is 10^{-5}.

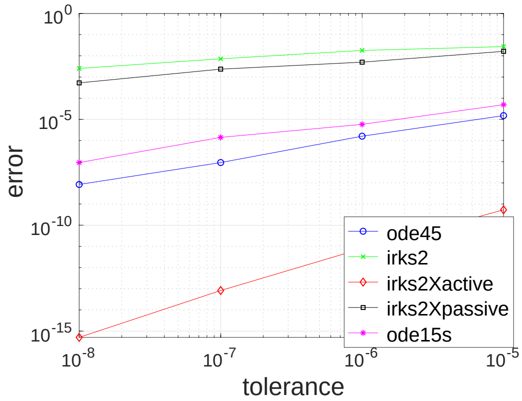

Figure 1. Numerical results for VDP equation using order-2 IRKS with passive and active extrapolations

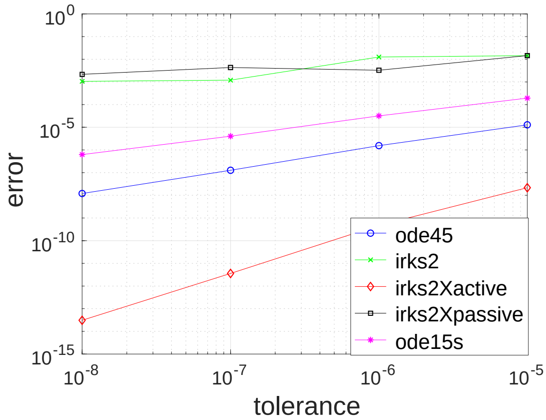

Figure 2. Numerical results for Bruss equation using order-2 IRKS with passive and active extrapolations

In the given numerical experiments, Figure 1 and Figure 2, we can conclude that the extrapolated IRKs methods are implemented more efficiently and accurately than the same methods without the extrapolation. Furthermore, active extrapolation gave the most accurate and efficient than others. Based on the above results, if we compare the current methods' efficiency with other famous methods such as ODE45 and ODE15s, we can still see the active extrapolation of general linear methods is super-efficient.

CONCLUSION AND FUTURE WORK

The numerical results show an improved accuracy for order 2 extrapolated IRKs methods to solve the non-stiff problems. For the test problems, order-2 general linear methods with extrapolation are efficient than that without extrapolation. Overall, the extrapolation improved the accuracy of the general linear methods with inherent Runge-Kutta stability. It is hoped that theoretical analysis of extrapolation for general linear methods with high orders can be given in the future.

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