ISSN 2462-2052
DOI: https://doi.org/10.37134/jsml.vol11.2.5.2023
Journal of Science and Mathematics Letters
Volume 11, Issue 2, 39-47, 2023

RESEARCH ARTICLE

Sequences of the Projection-valued Measures and Functional Calculi

Mykola Ivanovich Yaremenko*

Department of Partial Differential Equations, Faculty of Mathematics and Physics,
The National Technical University of Ukraine, 04213, Kyiv, Ukraine
Corresponding author: Math.kiev@gmail.com

Received: 3 March 2023; **Accepted:** 1 June 2023; **Published:** 16 June 2023

ABSTRACT

Let a pair (Φ, X) be functional calculus, where Φ is a homomorphism from the space of the measurable functions on (Ω, Σ) into the space of all linear bounded operators $LB(X, X)$ on a reflexive Banach space X. We define a norm of functional calculus Φ by $\|\Phi\| = \sup_{\|f\|=1,\ f \in \mathrm{M}(\Omega, \Sigma)} \|\Phi(f)\|_{L(X,X)}$, the convergence of the sequence of functional calculi is a convergence relative to this norm. We study the correspondence between sequences of spectral decompositions, well-bounded operators $\{A_n\}$ defined on the reflexive Banach space X, and their correspondence with the theory of functional calculus for such operators. In this article, we establish that if a sequence of the projection-valued measures $\{E_n(\lambda), \quad \lambda \in I\}$ strongly converges to $\{E(\lambda), \quad \lambda \in I\}$ then the sequence $\{\Phi_n\}$ of the functional calculi converges to the functional calculus Φ. Results of the article can be employed in the modern extensions of the quantum theory and theory of quantum information.

Keywords: functional calculus, projection-valued measure, projection operator, measurable calculus

1. INTRODUCTION

To the greatest extent, the concept of functional calculus generals the application of continuous functions to linear operators (Smart, 1959), which is especially important in the theory of partial differential equations, pseudo-differential operators, and their applications to models of quantum physics where the operator often is generated by Schrodinger equation. One of the possible approaches of functional calculus to the spectral theorem concerns the commuting normal operators on reflexive Banach spaces, which employs projection-valued measures (Haase, 2014). The spectral theorem is a crucial instrument of the operator theory and presents a straightforward way to formulate principal concepts of quantum mechanics in mathematical terms of operator theory, by postulating that the wave function of the quantum-mechanical system is governed by the Schrodinger equation. Another approach applies the apparatus of multiplication operators (Dungey, 2009). There is a fundamental correspondence between the functional calculi and the spectral representations of the operators on the reflexive Banach spaces. For instance, on any measurable space (Λ, Σ), a measurable functional calculus

Φ can be defined by the corresponding multiplication operator $M(\varphi)=\Phi(\varphi), \quad \varphi \in L(\Lambda,\Sigma)$, so for Hilbert space, H there is a well-known correlation between the measurable calculi and the operators of the multiplication that maintains the existence of the unitary operator $U: H \to L^2(\Omega,\wp,\mu)$ and an injective star-weak homomorphism $A: L(\Lambda,\Sigma) \to L(\Omega,\wp)$ such that $\Phi(f)=UM(A(f))U^{-1}$ for all $f \in L(\Lambda,\Sigma)$.

The functional calculus Φ is a pair (Φ, X), where Φ is a homomorphism from the space of the measurable functions defined on (Ω,Σ) into the space of the bounded linear operators $LB(X,X)$. A simple example is Borel functional calculus, which renders the possibility to extend concept functions of the real variable to include operators by prescription on the spectrum of the operators, for instance, the exponential or the resolvent of the Laplacian operator that appears in the Schrodinger equation. We can define a norm of functional calculus Φ as

$$\|\Phi\| = \sup_{\|f\|=1,\ f\in \mathrm{M}(\Omega,\Sigma)} \|\Phi(f)\|_{L(X,X)}$$

and consider the concept of convergence of functional calculi relative to this norm. The main result of this paper is an establishment that if a sequence of the projection-valued measures $\{E_n(\lambda), \quad \lambda \in I\}$ strongly converges to $\{E(\lambda), \quad \lambda \in I\}$ then the sequence $\{\Phi_n\}$ of the functional calculi converges to the functional calculus Φ.

There is extensive literature on the theory of functional calculi and projection-valued measures and their application to quantum problems of theoretical physics. These results are documented in the references.

2. MATERIALS AND METHODS

To address the equations of spectral theory, we employ generic function-analytic methods such as reflexiveness and convergence in the weak topology, methods Banach algebras and unital algebras homomorphisms; we also apply more specific methods of projection-valued measures, Stieltjes measures, Riesz–Markov–Kakutani representation approaches, and their combinations.

2.1. Convergence of the Sequence of the Projections-Valued Measures

Definition 1. A functional calculus Φ of the Banach algebra Θ of scalar-valued functions on Ω, for a bounded operator $A \in L(X,X)$, is a norm continuous algebra homomorphism $\Phi:\Theta \to B(X,X)$ such that $\Phi(z^n)=A^n$ for all natural indexes. If for all $x \in X$ morphism $\Phi_x:\Theta \to X$ defined the compact (weakly compact) operator as $\Phi_x(f)=\Phi(f)x$ then functional calculus Φ is called compact (weakly compact) functional calculus. As a consequence, let

$$p(z)=a_0+a_1 z+a_2 z^2+....+a_n z^n$$

be a polynomial over the complex field then the associated polynomial of the operator is

$$p(A)=a_0+a_1 A+a_2 A^2+....+a_n A^n.$$

The homomorphism Φ from the unital algebra of polynomials $\square(z)$ into the Banach algebra of the bounded operators $LB(X,X)$ is called a representation of the unital algebra

$\mathbb{C}(z)$ onto the Banach space X; namely operator $A \equiv \Phi(z)$ and $\Phi(p) = p(A)$ for all polynomials p defined on the complex plane $\mathbb{C}(z)$.

Definition 2. An operator-function $E(\lambda)$ is called a spectral decomposition of the operator A if the set $\{E(\lambda), \quad \lambda \in \mathbb{R}\} \subset LB(X, X)$ satisfies the following conditions

1. $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$ for $\lambda \le \mu$; and $\sup\limits_{\lambda} \|E(\lambda)\| < \infty$;
2. $E(\lambda) = strong - \lim\limits_{\lambda<\mu, \quad \mu \to \lambda} E(\mu)$;
3. $strong - \lim\limits_{\lambda \to -\infty} E(\lambda) = \mathrm{O}$ and $strong - \lim\limits_{\lambda \to \infty} E(\lambda) = I$;
4. $A = \int\limits_{\mathbb{R}} \lambda \, dE(\lambda) = strong - \lim\limits_{N \to \infty} \int\limits_{[-N, N]} \lambda \, dE(\lambda)$,

Where the integral is an operator-valued Riemann-Stieltjes integral in the topology of the operator norm.

Remark. Condition 1 means that the spectral decomposition is a bounded projection value measure.

Definition 3. Let f be a scalar-valued function on a compact interval $[a,b]$, and let us denote by $\Pi(\lambda) = \{a = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{n-1} < \lambda_n = b\}$ an arbitrary finite partition of the $[a,b]$, then variation of the function f over compact $[a,b]$ by definition is

$$\operatorname*{var}_{[a,b]} f = \sup_{\Pi(\lambda)} \sum_{i=1,\ldots,n} \left| f(\lambda_i) - f(\lambda_{i-1}) \right|.$$

If $\operatorname*{var}\limits_{[a,b]} f < \infty$, the function f is said to be of bounded variation over $[a,b]$. All functions of bounded variation comprise the Banach algebra $BV([a,b])$ with the natural norm $\|f\|_{BV([a,b])} = \operatorname*{var}\limits_{[a,b]} f$.

Definition 4. Let function $f \in BV([a,b])$ and let $\{E(\lambda), \quad \lambda \in R\}$ be a spectral decomposition concentrated on the compact interval $[a,b]$, the integral of the function $f \in BV([a,b])$ with respect to the spectral decomposition $\{E(\lambda)\}$ is

$$\overset{\oplus}{\int\limits_{[a,b]}} f(\lambda) dE(\lambda) = strong - \lim_{\lambda \in \Pi} \left(f(a)E(a) + \sum_{i=1,2,\ldots,n} f(\lambda_i)\left(E(\lambda_i) - E(\lambda_{i-1})\right) \right)$$

This definition is correct since assume $\{E(\lambda), \quad \lambda \in R\}$ is a spectral decomposition concentrated on the compact interval $[a,b]$ and $f \in BV([a,b])$, we have

$$f(a)E(a) + \sum_{i=1,2,\ldots,n} f(\lambda_i)\left(E(\lambda_i) - E(\lambda_{i-1})\right) =$$
$$= f(b)E(b) - \sum_{i=1,2,\ldots,n} \left(f(\lambda_i) - f(\lambda_{i-1})\right)E(\lambda_{i-1})$$

for any finite partition $\Pi(\lambda) = \{a = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{n-1} < \lambda_n = b\}$ of the compact interval $[a,b]$.

Theorem 1. Let f be a continuous function defined on $[a,b]$, and let a sequence $\{E_m(\lambda)\}$ of the spectral decompositions strongly converges to the spectral decomposition $E(\lambda)$ in each point of $[a,b]$. Then we have the limit

$$\lim_{m\to\infty} \int_{[a,b]} f(\lambda)\,dE_m(\lambda) = \int_{[a,b]} f(\lambda)\,dE(\lambda).$$

Proof. Let $\varepsilon > 0$ and let us split the compact interval $[a,b]$ into the finite subintervals as $a = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{n-1} < \lambda_n = b$ such that the fluctuation of the functions f on each interval $[\lambda_{i-1}, \lambda_i]$ is less than $\frac{\varepsilon}{3\mathrm{M}}$, where $M = \operatorname*{var}_{[a,b]}\left(\|E(\lambda)\|\right)$. Then

$$\int_{[a,b]} f(\lambda)\,dE(\lambda) = \sum_{i=1,2,\ldots,n} \int_{\lambda_{i-1}}^{\lambda_i} f(\lambda)\,dE(\lambda) =$$

$$= \sum_{i=1,2,\ldots,n} \int_{\lambda_{i-1}}^{\lambda_i} \left(f(\lambda) - f(\lambda_{i-1}) \right) dE(\lambda) +$$

$$+ \sum_{i=1,2,\ldots,n} f(\lambda_{i-1}) \int_{\lambda_{i-1}}^{\lambda_i} dE(\lambda),$$

where

$$\int_{\lambda_{i-1}}^{\lambda_i} dE(\lambda) = E(\lambda_i) - E(\lambda_{i-1}).$$

Since the fluctuation $\left| f(\lambda) - f(\lambda_{i-1}) \right|$ on the interval $[\lambda_{i-1}, \lambda_i]$ is less than $\frac{\varepsilon}{3M}$, we have

$$\left\| \sum_{i=1,2,\ldots,n} \int_{\lambda_{i-1}}^{\lambda_i} \left(f(\lambda) - f(\lambda_{i-1}) \right) dE(\lambda) \right\| \le \frac{\varepsilon}{3M} \operatorname*{var}_{[a,b]}\left(\|E(\lambda)\| \right).$$

So, there is $|\eta| \le 1$ such that

$$\left\| \int_{[a,b]} f(\lambda)\,dE(\lambda) - \sum_{i=1,2,\ldots,n} f(\lambda_{i-1}) \left(E(\lambda_i) - E(\lambda_{i-1}) \right) \right\| \le \eta \frac{\varepsilon}{3}.$$

Similarly, we can write

$$\int_{[a,b]} f(\lambda)\,dE_m(\lambda) = \sum_{i=1,2,\ldots,n} \int_{\lambda_{i-1}}^{\lambda_i} \left(f(\lambda) - f(\lambda_{i-1}) \right) dE_m(\lambda) +$$

$$+ \sum_{i=1,2,\ldots,n} f(\lambda_{i-1}) \int_{\lambda_{i-1}}^{\lambda_i} dE_m(\lambda)$$

and

$$\left\| \int_{[a,b]} f(\lambda)\,dE_m(\lambda) - \sum_{i=1,2,\ldots,n} f(\lambda_{i-1}) \left(E_m(\lambda_i) - E_m(\lambda_{i-1}) \right) \right\| \le \eta_m \frac{\varepsilon}{3},$$

where $|\eta_m| \le 1$ for all m.

Thus, for any $\varepsilon > 0$, there is a number $\tilde{m}$ such that the following inequality

$$\left\| \sum_{i=1,2,\ldots,n} f(\lambda_{i-1}) \left(E(\lambda_i) - E(\lambda_{i-1}) \right) - \sum_{i=1,2,\ldots,n} f(\lambda_{i-1}) \left(E_m(\lambda_i) - E_m(\lambda_{i-1}) \right) \right\| \le \frac{\varepsilon}{3}$$

holds for all $m > \tilde{m}$, thus for these m, we have

$$\left\| \int_{[a,b]} f(\lambda)\,dE(\lambda) - \int_{[a,b]} f(\lambda)\,dE_m(\lambda) \right\| \le \varepsilon,$$

the theorem has been proven.

Theorem 2. Let a sequence $\{A_n\}$ of the operators A_n converges to a well-bounded operator A and let a sequence $\{E_n(\lambda)\}$ of the spectral decompositions $E_n(\lambda)$ converges to the spectral decomposition $E(\lambda)$. Then

$$A = \lim_{n\to\infty} \overset{\oplus}{\int\limits_{[a,b]}} \lambda dE_n(\lambda) .$$

Proof. Since the operator A is well-bounded, there is a compact interval $[a,b]$ such that the following estimation

$$\|p(A)\| \le K\left(|p(b)| + \int\limits_{[a,b]} |p'(\lambda)| d\lambda\right)$$

holds for any polynomial p, so for large enough n, we have

$$\|p(A_n)\| \le K_n\left(|p(b)| + \int\limits_{[a,b]} |p'(\lambda)| d\lambda\right)$$

and thus operators A_n are well-bounded.

Since operators A_n and A are well-bounded, we can write the following representations

$$A_n = \overset{\oplus}{\int\limits_{[a,b]}} \lambda dE_n(\lambda) \text{ and } A = \overset{\oplus}{\int\limits_{[a,b]}} \lambda dE(\lambda) .$$

So, Theorem 1 implies Theorem 2.

Let us consider an example of the functional calculus of the entire functions. Let us represent an entire function $f \in Hol(\mathbb{C})$ as a power series in the form

$$f = \sum_{i=0,1,\dots} a_i z^i ,$$

where coefficients a_i are such that $\sum_{i=0,1,\dots} |a_i| r^i < \infty$ for all real numbers r. We can define unital algebras homomorphism

$$\Phi : Hol(\mathbb{C}) \to L(X, X), \ f \mapsto f(A)$$

as

$$\Phi(f) = f(A) = \sum_{i=0,1,\dots} a_i A^i \in L(X, X),$$

where $L(X, X)$ is unital Banach algebra over the reflexive Banach space X . The series $\sum_{i=0,1,\dots} a_i A^i$ converges absolutely in the $L(X, X)$ - norm. From the Cauchy formula

$$\left(\sum_{i=0,1,\dots} a_i\right)\left(\sum_{i=0,1,\dots} b_i\right) = \sum_{i=0,1,\dots}\left(\sum_{j=0,1,\dots,i} a_j b_{i-j}\right)$$

follows the multiplicativity of functional calculus.

Thus, functional calculus Φ defined correspondence between the functional space $Hol(\mathbb{C})$ and the unital Banach algebra $L(X, X)$. Now, assuming this correspondence depends on the natural parameter n, then we have the sequence of the functional calculi Φ_n. Continue our example, the sequence $\{\Phi_n\}$ can be defined as

$$\Phi_n(f) = \sum_{i=0,1,\dots} a_i n^{-i} A^i \in L(X, X)$$

for $n > 0$.

2.2. *Convergence of the Sequence of the Functional Calculi*

Let (Ω, Σ) be a measurable space and $M(\Omega, \Sigma)$ be a measurable functional space over (Ω, Σ).

Definition 5. A pair (Φ, X) is a functional calculus on a measurable space (Ω, Σ, μ), where X is a reflexive Banach space, and a morphism Φ maps measurable space $\mathrm{M}(\Omega, \Sigma)$ into the set of closed operators $L(X, X)$. The morphism Φ must satisfy the following conditions:

1. $\Phi(\mathbf{1}) = I$;
2. $\alpha\Phi(f) + \beta\Phi(g) \subseteq \Phi(\alpha f + \beta g)$ *for all* $f, g \in \mathrm{M}(\Omega, \Sigma)$ *and all* $\alpha, \beta \in P$;
3. $\Phi(f)\Phi(g) \subseteq \Phi(fg)$ *and* $\mathrm{dom}(\Phi(f)\Phi(g)) = \mathrm{dom}(\Phi(g)) \cap \mathrm{dom}(\Phi(fg))$ *for all* $f, g \in \mathrm{M}(\Omega, \Sigma)$;
4. $\Phi(f) \in L(X, X)$;
5. *if* $\|f_n - f\| \to 0$ *and* $\|f_n - f\|_\infty \to 0$ *then* $weak - \lim_{n\to\infty} \Phi(f_n) = \Phi(f)$,

where P denotes the scalar field.

Let us define a real-valued function of functional calculus Φ as

$$\|\Phi\| = \sup_{f \in \mathrm{M}(\Omega, \Sigma)} \frac{\|\Phi(f)\|_{L(X,X)}}{\|f\|},$$

such a defined function satisfies all requirements to be a norm of the functional calculus Φ.

Definition 6. Functional calculus Φ is called the strong limit of the sequence $\{\Phi_n\}$ of the functional calculi if for any $\varepsilon > 0$ there is a number $\tilde{n}(\varepsilon)$ such that inequality

$$\|\Phi - \Phi_n\| < \varepsilon$$

holds for all $n > \tilde{n}(\varepsilon)$.

Let $\{E_n(\lambda), \quad \lambda \in I\}$ be a sequence of the projection-valued measures, which means that, for each n, projection E_n maps σ-algebra Σ into $L(X, X)$; we denote $AC([a,b])$ a space of absolutely continuous functions on the compact interval $[a,b]$.

Theorem 3. Let $\{E_n(\lambda), \quad \lambda \in I\}$ be a sequence of the projection-valued measures and let $\{E_n(\lambda), \quad \lambda \in I\}$ strongly converge to $\{E(\lambda), \quad \lambda \in I\}$ then the sequence $\{\Phi_n\}$ of the functional calculi converges to the functional calculus Φ.

Proof. For any fixed number n the projection-valued measure $\{E_n(\lambda), \quad \lambda \in I\}$ defines the map $\Phi_n : AC([a,b]) \to LB(X, X)$ by the formula

$$\Phi_n(f) = \int_{[a,b]}^{\oplus} f(\lambda)\, dE_n(\lambda)$$

for every function $f \in AC([a,b])$. The maps Φ_n are linear and bounded, and such that

$$\Phi_n(f \mapsto 0) = \int_{[a,b]}^{\oplus} 0\, dE_n(\lambda) = \mathrm{O},$$

$$\Phi_n(f \mapsto 1) = \int_{[a,b]}^{\oplus} dE_n(\lambda) = \mathrm{I}.$$

ISSN 2462-2052 Journal of Science and Mathematics Letters
DOI: https://doi.org/10.37134/jsml.vol11.2.5.2023 Volume 11, Issue 2, 39-47, 2023

For each fixed $x \in X, x^* \in X^*$ the maps $\lambda \mapsto E_n(\lambda)x$ are right continuous since

$$\left\langle \Phi_n(\lambda \mapsto \lambda)x, x^* \right\rangle = \overset{\oplus}{\int\limits_{[a,b]}} \lambda d\left\langle E_n(\lambda)x, x^* \right\rangle =$$

$$= b\left\langle x, x^* \right\rangle - \int\limits_{[a,b]} \left\langle E_n(\lambda)x, x^* \right\rangle d\lambda.$$

Similarly, the map $\Phi : AC([a,b]) \to LB(X, X)$ is defined by

$$\Phi(f) = \overset{\oplus}{\int\limits_{[a,b]}} f(\lambda) dE(\lambda) .$$

Thus, the maps Φ_n, Φ are functional calculi.

Assume $\varepsilon > 0$ and let us divide the interval $[a,b]$ into the subintervals as $a = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{m-1} < \lambda_m = b$ such that the fluctuation of the functions $f \in AC([a,b])$ on each interval $[\lambda_{i-1}, \lambda_i]$ is less than $\frac{\varepsilon}{3M}$, where $M = \sup_n \left(\operatorname{var}_{[a,b]}(\|E_n(\lambda)\|), \operatorname{var}_{[a,b]}(\|E(\lambda)\|) \right)$.

Then

$$\int\limits_{[a,b]} f(\lambda) dE(\lambda) = \sum_{i=1,2,\ldots,m} \int\limits_{\lambda_{i-1}}^{\lambda_i} \left(f(\lambda) - f(\lambda_{i-1}) \right) dE(\lambda) +$$

$$+ \sum_{i=1,2,\ldots,m} f(\lambda_{i-1}) \int\limits_{\lambda_{i-1}}^{\lambda_i} dE(\lambda),$$

and

$$\int\limits_{[a,b]} f(\lambda) dE_n(\lambda) = \sum_{i=1,2,\ldots,m} \int\limits_{\lambda_{i-1}}^{\lambda_i} \left(f(\lambda) - f(\lambda_{i-1}) \right) dE_n(\lambda) +$$

$$+ \sum_{i=1,2,\ldots,n} f(\lambda_{i-1}) \int\limits_{\lambda_{i-1}}^{\lambda_i} dE_m(\lambda),$$

so, there is $|\eta| \le 1$ and $|\eta_m| \le 1$ such that

$$\left\| \int\limits_{[a,b]} f(\lambda) dE(\lambda) - \sum_{i=1,2,\ldots,m} f(\lambda_{i-1}) \left(E(\lambda_i) - E(\lambda_{i-1}) \right) \right\| \le \eta \frac{\varepsilon}{3}.$$

and

$$\left\| \int\limits_{[a,b]} f(\lambda) dE_m(\lambda) - \sum_{i=1,2,\ldots,m} f(\lambda_{i-1}) \left(E_n(\lambda_i) - E_n(\lambda_{i-1}) \right) \right\| \le \eta_n \frac{\varepsilon}{3}.$$

For any $\varepsilon > 0$, there is a number $\tilde{n}$ such that

$$\left\| \int\limits_{[a,b]} f(\lambda) dE(\lambda) - \int\limits_{[a,b]} f(\lambda) dE_n(\lambda) \right\| \le \varepsilon$$

holds for all $n > \tilde{n}$.

Thus, we have obtained that for any $\varepsilon > 0$, there is a number $\tilde{n}(\varepsilon)$ such that $\|\Phi - \Phi_n\| \le \varepsilon$ holds for every $n > \tilde{n}(\varepsilon)$. The theorem is proven.

3. RESULTS AND DISCUSSION

We call a pair (Φ, X) a functional calculus defined on space (Ω, Σ, μ) if the morphism Φ such that $\Phi(\mathbf{1}) = I$; $\alpha\Phi(f) + \beta\Phi(g) \subseteq \Phi(\alpha f + \beta g)$ for all $f, g \in \mathrm{M}(\Omega, \Sigma)$ and all $\alpha, \beta \in P$; $\Phi(f)\Phi(g) \subseteq \Phi(fg)$ and $\mathrm{dom}(\Phi(f)\Phi(g)) = \mathrm{dom}(\Phi(g)) \cap \mathrm{dom}(\Phi(fg))$ for all $f, g \in \mathrm{M}(\Omega, \Sigma)$; $\Phi(f) \in L(X, X)$; form $\|f_n - f\| \to 0$ and $\|f_n - f\|_\infty \to 0$ follows that $\lim_{n\to\infty} \Phi(f_n) = \Phi(f)$ in the weak topology. In this paper, we establish that assume $\{E_n(\lambda), \quad \lambda \in I\}$ is a sequence of the projection-valued measures, where $AC([a,b])$ is a space of absolutely continuous functions on the compact interval $[a,b]$, then from $\{E_n(\lambda), \quad \lambda \in I\}$ strongly converges to $\{E(\lambda), \quad \lambda \in I\}$ follows that the sequence $\{\Phi_n\}$ of the functional calculi converges to the functional calculus Φ. Interesting example of an unbounded operator is operator Volterra, which corresponds with the Riesz calculus and generates the Riemann-Louville semigroup.

4. CONCLUSION

Assume a sequence $\{A_n\}$ converges to a well-bounded operator A and the spectral decompositions sequence $\{E_n(\lambda)\}$ converges to the spectral decomposition $E(\lambda)$, then the operator A can be presented in the form $A = \lim_{n\to\infty} \int_{[a,b]}^{\oplus} \lambda dE_n(\lambda)$. The sequence $\{E_n(\lambda), \quad \lambda \in I\}$ of the projection-valued measures, which strongly converge to the projection-valued measure $\{E(\lambda), \quad \lambda \in I\}$ corresponds to the functional calculus Φ, which is the strong limit of the sequence $\{\Phi_n\}$ of the functional calculi Φ_n corresponded to $E_n(\lambda)$. Additional studies are needed to develop these results on more general classes of operators and topological spaces.

Declaration of Interest
The author declares that there is no conflict of interest.

REFERENCES

Alfeus M., Schlogl E. (2018). On numerical methods for spread options. Quantitative Finance Research Centre. http://www.qfrc.uts.edu.au.

Arendt W, Vogt H, Voigt J. (2019). Form Methods for Evolution Equations. Lecture Notes of the 18th International Internet seminar, version: 6 March.

Batty C, Gomilko A, Tomilov Y. (2015). Product formulas in functional calculi for sectorial operators. *Mathematische Zeitschrift*, 279, 479-507.

Budde C, Landsman K. (2016). A bounded transform approach to self-adjoint operators: functional calculus and affiliated von Neumann algebras. *Annals of Functional Analysis*, 7, 411-420.

Colombo F, Gentili, G, Sabadini I, Struppa DC. (2010). Non-commutative functional calculus: Unbounded operators. *Journal of Geometry and Physics*, 60(2), 251-259.

Colombo F, Sabadini I. (2009). On Some Properties of the Quaternionic Functional Calculus. *Journal of Geometric Analysis*, 19, 601.

DeLaubenfels R. (1995). Automatic extensions of functional calculi. *Studia Mathematica*, 114, 237-259.

Dungey N. (2009). Asymptotic type for sectorial operators and an integral of fractional powers. *Journal of Functional Analysis*, 256, 1387-1407.

Eisner T, Farkas B, Haase M, Nagel R. (2015). Operator theoretic aspects of ergodic theory. Vol. 272 of Graduate Texts in Mathematics. Springer, Cham.

Florentino C, Nozad A, Zamora A. (2020). Serre polynomials of S Ln - and PG Ln-character varieties of free groups, *Journal of Geometry and Physics*, 161, 104008.

Haase M. (2014). Functional analysis. An Elementary Introduction. Vol. 156 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.

Mhlanga FJ. (2015). Calculations of Greeks for jump-diffusion Processes. *Mediterranean Journal of Mathematics,* 12, 1141-1160.

Ringrose J. (1960). On well-bounded operators. *Journal of the Australian Mathematical Society*, 1(3), 334-343.

Schmudgen K. (2012). Unbounded Self-adjoint Operators on Hilbert Space. Vol. 265 of Graduate Texts in Mathematics. Springer, Dordrecht.

Smart DR. (1959). Eigenfunction expansions in Lp and C, *Illinois Journal of Mathematics* 3, 82-97.

Van Belle J, Vanduffelm S, Yao J. (2019). Closed-form approximations for spread options in Levy markets. *Applied Stochastic Models in Business and Industry*, 35, 732-746.

Yaremenko MI. (2012). Calderon-Zygmund operators and singular integrals. *Applied Mathematics & Information Sciences*, 15, 97-107.

Yilmaz B. (2018). Computation of option Greeks under hybrid stochastic volatility models via Malliavin calculus. *Modern Stochastics: Theory and Applications*, 5(2), 145-165.